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A TREATISE

ON THE

MATHEMATICAL THEORY

OF

ELASTICITY.

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A TREATISE

ON THE

MATHEMATICAL THEORY

OF

ELASTICITY

BY

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VOLUME I.

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PREFACE.

THE present treatise is the outcome of a suggestion made to me some years ago by Mr R. R. Webb that I should assist him in the preparation of a work on Elasticity. He has unfortunately found himself unable to proceed with it, and I have therefore been obliged to take upon myself the whole of the work and the whole of the responsibility. I wish to acknowledge at the outset the debt that I owe to him as a teacher of the subject, as well as my obligation for many valuable suggestions chiefly with reference to the scope and plan of the work, and to express my regret that other engagements have prevented him from sharing more actively in its production.

The division of the subject adopted is that originally made by Clebsch in his classical treatise, where a clear distinction is drawn between exact solutions for bodies all whose dimensions are finite and approximate solutions for bodies some of whose dimensions can be regarded as infinitesimal. The present volume contains the general mathematical theory of the elastic properties of the first class of bodies, and I propose to treat the second class in another volume. At Mr Webb's suggestion, the exposition of the theory is preceded by an historical sketch of its origin and development. Anything like an exhaustive history has been rendered unnecessary by the work of the late Dr Todhunter as edited by Prof. Karl Pearson, but it is hoped that the brief account given will at once facilitate the comprehension of the theory and add to its interest.

Readers of the historical work referred to will appreciate the difficulty of giving within a reasonable compass a complete account of all the valuable researches that have been made; and the aim of this book is rather to present a connected account of the theory in its present state, and an indication of the way in which that state has been attained, avoiding on the one hand merely analytical developments, and on the other purely technical details.

The first five chapters are occupied with the general theory, including the analysis of strain and stress, stress-strain relations, the strength of materials, and a number of general theorems. In the analysis of strain I have thought it best to follow Thomson and Tait's *Natural Philosophy*, beginning with the geometrical or rather algebraical theory of finite homogeneous strain, and passing to the physically most important case of infinitesimal strain. In the deduction of the general equations of equilibrium or small motion I have set out from the equations that must be satisfied by a finite portion of the mass. The discussion of the stress-strain relations rests upon Hooke's Law as an axiom generally verified in experience, and on Sir W. Thomson (now Lord Kelvin's) thermodynamical investigation of the existence of the energy-function. To understand the work that has been done upon æolotropic bodies requires some knowledge of Crystallography, and a short sketch of that subject is given. The theory of elastic crystals adopted is that which has been elaborated by the researches of F. E. Neumann and W. Voigt. To understand the nature of the application of the theory of elasticity to practical problems it is necessary to have some knowledge of the behaviour of bodies more than infinitesimally strained, and I have given a short sketch of what is known in regard to technical elasticity. The conditions of rupture or rather of safety of materials are as yet so little understood that it seemed best to give a statement of the various theories that have been advanced without definitely adopting any of them. In most of the problems considered in the text Saint-Venant's "greatest strain" theory has been provisionally adopted. In connexion with this theory I have endeavoured to give precision

to the term "factor of safety". Among general theorems I have included an account of the deduction of the theory from Boscovich's point-atom hypothesis. This is rendered necessary partly by the controversy that has raged round the number of independent elastic constants, and partly by the fact that there exists no single investigation of the deduction in question which could now be accepted by mathematicians.

Chapter VI. treats of Saint-Venant's theory of the equilibrium of beams. In spite of the work of Prof. Pearson it seems not yet to be understood by English mathematicians that the cross-sections of a bent beam do not remain plane. The old-fashioned notion of a bending moment proportional to the curvature resulting from the extensions and contractions of the fibres is still current. Against the venerable bending moment the modern theory has nothing to say, but it is quite time that it should be generally known that it is not the whole stress, and that the strain does not consist simply of extensions and contractions of the fibres. In explaining the theory I have followed Clebsch's mode of treatment, generalising it so as to cover some of the classes of æolotropic bodies treated by Saint-Venant.

Chapter VII. contains a short account of the theory of curvilinear coordinates with applications to Elasticity. I regret that the theory was written out before I had seen M. Ossian Bonnet's researches on the subject, in which the kinematical method adopted was largely anticipated.

The remaining chapters are occupied with the principal analytical problems presented by elastic theory. The theory leads in every special case to a system of partial differential equations, and the solution of these subject to conditions given at certain bounding surfaces is required. The general problem is that of solving the general equations with arbitrary conditions at any given boundaries. In discussing this problem I have made extensive use of the researches of Prof. Betti of Pisa, whose investigations are the most general that have yet been given, and appear to admit of considerable further development. The case of

a solid bounded by an infinite plane and otherwise unlimited is investigated on the lines laid down by Signor Valentino Cerruti, whose analysis is founded on Prof. Betti's general method, and some of the most important particular cases are worked out synthetically by M. Boussinesq's method of potentials. In this connexion I have introduced the last-mentioned writer's theory of "local perturbations", a theory which gives the key to Saint-Venant's "principle of the elastic equivalence of statically equipollent systems of load". The classical problems of the equilibrium and vibrations of a sphere, with applications to tidal and other problems connected with the Earth, are investigated by the methods of Lord Kelvin and Prof. Lamb. I believe that the use of Cartesian coordinates in these problems at once shortens and simplifies the work. In the last chapter a few further examples of the solution of the general equations are given. Although so much space is devoted to analytical discussions I venture to hope that the problems selected for treatment will be found to be those that possess the greatest physical interest, and I consider a treatise on the mathematical theory of Elasticity would be incomplete if it gave no account of the principal mathematical problem associated with the subject.

There are some matters treated by elasticians which I have omitted. Among these are thermo-elasticity, photo-elasticity, and the elastic solid theory of Optics. None of these subjects are so satisfactory either in their data or in their conclusions as the part of the theory selected, viz. the rational mechanics of Elasticity.

The choice of a suitable notation has been a matter of considerable difficulty. In this I have been guided partly by some remarks in a paper of Prof. Lamb's, and partly by the experience that there is much less difficulty in mentally associating a simple and unsuggestive notation with a cumbersome and suggestive one than in using the latter.

The references given at the heads of most of the chapters are generally to the sources from which parts of the work are taken, but occasionally they include also investigations which follow

totally different methods from those given in the text. They are intended as an acknowledgement of indebtedness, and a suggestion to the reader for further work. I have tried to avoid as far as possible reference either to erroneous mathematics or to inconclusive experiments.

I have not thought it advisable to introduce collections of examples for practice. On the other hand a number of results are stated without proof. These are generally either of historical interest or else of importance in the development of the subject, but the analysis necessary to prove them would involve no point but such as will be found in the text, or may fairly be assumed to be known. The student without previous acquaintance with the subject is advised in all cases to provide the required proofs. It is hoped that he will not then fail to understand the subject for lack of examples, nor waste his time in mere problem grinding.

In conclusion I have to express my thanks to Prof. A. G. Greenhill, Prof. Karl Pearson, and Mr J. Larmor for their kindness in reading the proof-sheets and for many valuable criticisms, to Mr A. Harker for his kind assistance in the revision of the articles on Crystallography, and to Mr C. Chree for his very careful revision of the proofs, and for the many suggestions he has made for the improvement of the work during its passage through the press.

A. E. H. LOVE.

St John's College, Cambridge
April, 1892.

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CORRIGENDA.

p. 106, ft. note 5, *for* art. 150 *read* art. 130.

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HISTORICAL INTRODUCTION.

THE mathematical theory of Elasticity is occupied with an attempt to reduce to calculation the state of strain, or relative displacement, within a solid body which is subject to the action of an equilibrating system of forces, or is in a state of small internal relative motion, by the aid of experimental data and physical axioms assumed in advance, and with endeavours to obtain results which shall be practically important in applications to architecture, engineering, and all other useful arts in which the material of construction is solid. Its history should embrace that of the progress of our experimental knowledge of the behaviour of strained bodies, so far as it has been embodied in the mathematical theory, of the development of our conceptions in regard to the physical axioms necessary to form a foundation for theory, of the growth of that branch of mathematical analysis in which the process of the calculations consists, and of the gradual acquisition of practical rules by the interpretation of analytical results. We propose to give a sketch of such a history, so far as to include the subject-matter of the present volume, excluding the special problems of the equilibrium and vibrations of thin wires and plates, and the related theories of impact and elastic stability. In a subject ideally worked out, the progress which we should be able to trace would be, in other particulars, one from less to more, but we may say, that in regard to the assumed physical axioms, progress consists in passing from more to less. Alike in the experimental knowledge obtained, and in the analytical methods and results, nothing that has once been discovered ever loses its value, or has to be discarded; but the physical axioms come to be reduced to fewer and more general principles, so that the theory is brought more into accord with that of other physical subjects, the

same general dynamical principles being ultimately requisite and sufficient to serve as a basis for them all. And although, in our subject, we find frequent retrogressions on the part of the experimentalist, and errors on the part of the mathematician, chiefly in adopting hypotheses not clearly established or already discredited, in pushing to extremes methods merely approximate, in hasty generalisations, and in misunderstandings of physical principles, yet we observe a steady and continuous progress in all the respects mentioned when we survey the history of our subject from the first enquiries of Galilei to the final works of Saint-Venant and Sir William Thomson.

The first mathematician to consider the nature of the resistance of solids to rupture was Galilei[1]. Although he treated solids as inelastic, not being in possession of any law connecting the displacements produced with the force producing them, or of any physical hypothesis capable of yielding such a law, yet his enquiries gave the direction which was subsequently followed by many investigators. He endeavoured to determine the resistance of a beam, one end of which is built into a wall, when the tendency to break it arises from its own or an applied weight, and he concluded that the beam tends to turn about an axis perpendicular to its length, and in the plane of the wall. This problem, and, in particular, the determination of this axis is known as Galilei's problem.

In the history of the theory started by the question of Galilei, undoubtedly the two great landmarks are the discovery of Hooke's Law in 1660, and the discovery of the general equations by Navier in 1821. The first gave the fundamental experimental datum, required for the foundation of the theory, the second reduced all questions of the small strain of elastic bodies to a matter of mathematical calculation.

In England and in France, in the latter half of the 17th century, Hooke and Marriotte occupied themselves with the experimental discovery of what we now term stress-strain relations. Hooke gave in 1678[2] the famous law of proportionality of stress and strain which bears his name, in the words "*Ut tensio sic vis;* that is, the Power of any spring is in the same proportion with

[1] See Todhunter and Pearson's *History*, vol. I. The date of Galilei's enquiry is 1638.

[2] In his work *De Potentia Restitutiva*. London, 1678.

the Tension thereof". By "spring" Hooke means, as he proceeds to explain, any "springy body", and by "tension" what we should now call "extension", or, more generally, "strain". This law he discovered in 1660, but did not publish until 1676, and then only under the form of an anagram *ceiiinosssttuu*. This law forms the basis of the mathematical theory of Elasticity, and we shall hereafter have to consider its generalisation, and its present position in the light of modern experimental research. Hooke does not appear to have made any application of it to the consideration of Galilei's problem. This was reserved for Marriotte[1], who in 1680 made the same experimental discovery. He remarked that the resistance of a beam to flexure arises from the extension and compression of its parts, some of its fibres being extended, and others compressed. He assumed that half are extended, and half compressed. His theory led him to assign the position of the axis, required in the solution of Galilei's problem, at one half the height of the section above the base.

In the interval between the discovery of Hooke's law, and that of the general differential equations of elasticity by Navier, the attention of those mathematicians who occupied themselves with our subject was chiefly directed to the solution and extension of Galilei's problem, and the analogous theories of the vibrations of bars, and the stability of columns. The first investigation of any importance is that of the elastic line by James Bernoulli[2] in 1705, in which the resistance of a bent rod is assumed to arise from the extension and compression of its fibres, and the equation of the curve assumed by the axis is formed. The equation of the axis practically involves that the stress across any section reduces to a couple proportional to the curvature. This was expressly or practically assumed (not proved) by Euler[3] and Daniell Bernoulli[4] in their later treatment of the related problem of the vibrations of bars. It would carry us too far into the history of special problems to give a detailed account of the memoirs of this period on this subject, but Prof. Pearson's remarks[5] on the quasi-theologi-

[1] *Traité du mouvement des eaux.* Paris, 1686.

[2] Bernoulli's memoir is entitled 'Veritable Hypothése de la Résistance des Solides, avec la demonstration de la courbure des corps qui font ressort', and will be found in his collected works, vol. II. Geneva, 1744.

[3] *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes.*

[4] See in particular his letter of Oct. 1742, art. 46.

[5] *History*, vol. I. p. 34.

cal character of the arguments usually employed to reduce a dynamical problem to mathematical analysis, will be read with great interest by all those who study the history of the development of human thought. Of more importance for our present purpose is Coulomb's[1] theory of flexure, given in what must be regarded as the most scientific of all the early mathematical memoirs dealing with Galilei's problem. This author took account of the equation of equilibrium obtained by resolving horizontally the forces, which act upon the part of the beam cut off by one of its normal sections, as well as of the equation of moments. This enabled him to obtain the true position of the "neutral line", or axis of equilibrium, and he also made a correct calculation of the moment of the elastic forces. His theory of beams is the most exact, that proceeds on the assumption that the stress in a bent beam arises wholly from the extension and compression of its fibres, and is deduced mathematically from this assumption and Hooke's Law. Coulomb was also the first to consider the resistance of thin fibres to torsion[2], and it is his account of the matter to which Saint-Venant refers under the name *l'ancienne théorie*, but his formula for this resistance was not deduced from any elastic theory. The formula makes the torsional rigidity of a fibre proportional to the moment of inertia of the normal section about the axis of the fibre. Another matter to which Coulomb was the first to pay attention was the kind of strain we now call *shear*, though he only considered it in connexion with rupture. His opinion appears to have been that rupture[3] takes place, when the shear of the material is greater than a certain limit. The shear considered is a permanent set, not an elastic strain.

Except Coulomb's the most important work of the period, for the general mathematical theory of elasticity, is the physical consideration of the subject by Thomas Young. This naturalist, (to adopt Sir William Thomson's name for students of natural science,) besides defining his modulus of elasticity, was the first to consider shear[4] as an elastic strain. He called it "detrusion", and

[1] 'Essai sur une application des règles *de Maximis et Minimis* à quelques Problèmes de Statique, relatifs à l'Architecture'. *Mém....par divers savans*, 1776, pp. 350—354.

[2] *Histoire de l'Académie* for 1784, pp. 229—269, Paris, 1787.

[3] See the memoir first quoted, *Mém....par divers savans*, Introduction.

[4] *A course of lectures on Natural Philosophy and the Mechanical Arts*, 1807, Lecture III. It is in Kelland's later edition (1845) on pp. 105 sq.

noticed that the elastic resistance of a body to shear, and its resistance to extension or compression, are in general different; but he did not expressly introduce a new modulus of rigidity for this resistance. He defined "the modulus of elasticity of a substance[1]" as a column of the substance capable of producing a pressure on its base, which is to the weight causing a certain degree of compression, as the length of the substance is to the diminution of its length. What we now call "Young's modulus" is the weight of this column per unit of area of its base. This introduction of a definite physical concept, associated with the coefficient of elasticity, which descends as it were from a clear sky on the reader of mathematical memoirs, marks an epoch in the history of the science.

In the literature of this, the first period in the history of our subject, there are many discussions of the physical cause of elasticity, the philosophers, generally, either following Descartes, and believing in space continuously filled and a subtle æther that is in the pores of bodies, or else following the suggestion of Newton, that all the interactions between parts of bodies can be reduced to attracting and repelling forces between the ultimate molecules, which operate immediately, without any intervening mechanism. But no attempt appears to have been made to deduce general equations of motion and equilibrium from either of these hypotheses. At the end of the year 1820, the fruit of all the ingenuity expended on elastic problems might be summed up as—an inadequate theory of flexure, an erroneous theory of torsion, an unproved theory of the vibrations of bars, and the definition of Young's modulus. But such an estimate would give a very wrong impression of the value of the older researches. The recognition of the fact, that there is a fundamental difference between shear and extension, was a preliminary to a general theory of strain; the discovery of forces across a section of a beam, producing a resultant, was a step towards a theory of stress; the use of differential equations for the deflexion of a bent beam and the vibrations of bars and plates, was a foreshadowing of the employment of differential equations of displacement; the suggestion of Newton and the enunciation of Hooke's law, offered means for the formation

[1] *Loc. cit.* This was given in section IX. of vol. II. of the first edition, and omitted in Kelland's edition, but it is reproduced in the *Miscellaneous Works* of Dr Young.

of such equations; and the generalisation of the principle of virtual work in the *Mécanique Analytique* threw open a broad path to discovery in this as in every other branch of mathematical physics. Physical Science had emerged from its incipient stages with definite methods, of hypothesis and induction and of observation and deduction, with a clear aim, to explain facts, and with a fund of analytical processes of investigation. This was the hour for the production of general theories, and the men were not wanting.

There are two subjects, usually included in the general theory of elasticity, which have an extended application to other branches of mathematical Physics, these are the analysis of strain and the analysis of stress. The first gives general considerations as to the kinematical expression of the possible deformations of the parts of any medium which can be treated as continuous, the second gives similar considerations relative to the kind of internal forces that can exist in such media. The foundation of both theories was laid by Cauchy in 1827, but he appears to have been in possession of some of the results as early as 1822, when he communicated an account of his researches to the Paris Academy[1]. Among his discoveries[2] must be reckoned the determination of the stress at any point in terms of six[3] component stresses, and of the strain, whether finite or infinitesimal, in terms of six component strains, the properties of the stress-quadric, stress-ellipsoid, strain-quadric, and elongation-quadric, and the existence of principal stresses and principal extensions. Results equivalent to some of Cauchy's were discovered independently by Lamé[4], who developed somewhat the geometrical study of distributions of stress by means of the properties of certain quadric surfaces. Cauchy's expressions of the six com-

[1] *Bulletin...Philomatique*, 1823.

[2] See *Exercices de Mathématiques*, 1827, in which are the following memoirs: 'De la pression ou tension dans un corps solide', 'Sur la condensation et la dilatation des corps solides', and *Exercices de Mathématiques* 1828, in which is a memoir 'Sur quelques theorèmes relatifs à la condensation ou à la dilatation des corps'.

[3] The assumption involved in this reduction does not appear to have been noticed by writers on elastic theory. The fact that a medium is possible in which it does not hold good appears to have been first noticed in connexion with Electrodynamics.

[4] Lamé and Clapeyron, 'Mémoire sur l'équilibre intérieur des corps solides homogènes'. *Mém....par divers savans*, IV. 1833. The date of the memoir is at least as early as 1828.

ponents of finite strain are practically those of Green[1], and Saint-Venant[2], but the latter was the first to consider them minutely. To Saint-Venant more than anyone else belongs the credit of the adequate discussion of shear[3]; he was the first mathematician to call attention to its importance as a specific kind of strain; previously to his time the quantities we should now call shears made their appearance simply as mathematical expressions. Sir W. Thomson further simplified the discussion of strain by the introduction of his strain-ellipsoid[4], and the kinematical theory reaches its highest development in Thomson and Tait's *Natural Philosophy*, Part I.

To a modern reader it might appear that the analysis of stress and strain is a necessary preliminary to a general theory of elasticity, but historically this was not the order in which discoveries were made. The investigation of the general equations by Navier does not depend on any such analysis; Poisson's investigation involves an analysis of stress, but not of strain, Green's an analysis of strain, but not of stress. There are in fact three fundamental methods of arriving at these equations. The first consists in assuming a law as to the character of intermolecular force, and deducing the differential equations of displacement from the equations of equilibrium of a single displaced "molecule". This is Navier's method. The second method consists in forming differential equations of equilibrium of any element in terms of the stresses exerted upon it by the surrounding matter, and then, by means of relations between stress and relative displacement, eliminating the stress-components from these equations. The required relations may be assumed, as in Cauchy's first investigation, or deduced from experiment, as by Sir G. Stokes, or calculated from an assumed law of intermolecular force, as by Poisson and Cauchy. The third method consists in writing down an expression for the energy of the strained solid, and deducing

[1] 'On the Laws of Reflexion and Refraction of Light at the common surface of two non-crystallized media', *Camb. Phil. Soc. Trans.* VII. 1837. See also *Math. Papers of the late George Green*, 1871.

[2] 'Mémoire sur l'équilibre des corps solides', *Comptes rendus*, XXIV. 1847. The expressions referred to were given by Saint-Venant in 1844, see Todhunter and Pearson, vol. I. art. 1614.

[3] *Leçons de Mécanique appliquée*, 1837, 1838. See Todhunter and Pearson, vol. I. arts. 1564, 1565, 1570.

[4] Thomson and Tait, *Nat. Phil.* Part I. arts. 155—190.

the equations by an application of the principle of Virtual Work. This method is due to Green, and has been followed by Kirchhoff and many English writers.

Navier[1] was the first to investigate the general equations of equilibrium and vibrations of elastic solids. He set out from the hypothesis which we have ascribed to Newton, that the elastic reactions arise from variations in the intermolecular forces, consequent upon changes in the molecular configuration. He assumed that the force between two molecules, whose distance is slightly increased, is proportional to the product of the increase in the distance and some function of the initial distance. His method consists in forming an expression for the component in any direction of all the forces, that act upon a displaced "molecule", and thence the equations of motion of the molecule. The equations are thus obtained in terms of the displacements of the molecule. The solid is assumed to be isotropic, and the equations obtained contain a single elastic constant. Navier next formed an expression for the work done in a small relative displacement by all the forces which act upon a molecule; this he described as the sum of the moments in the sense of the *Mécanique Analytique* of the forces exerted by all the other molecules on a particular molecule. He deduced, by an application of the Calculus of Variations, not only the differential equations previously obtained, but also the boundary-conditions that hold at the surface of the body. This memoir is very important as the first general investigation of its kind, but its arguments would not now be admitted. In the first place the expression for the force between two molecules, after displacement, is incorrect; in the second place the expression for the component force in any direction, acting on a molecule, is wrongly discussed[2]. This expression involves a triple summation, and Navier replaced the summations by integrations. It appears from subsequent investigations by Cauchy and Poisson that this step is unnecessary, and, if the force between two molecules be taken simply a function of their distance, it leads to absurd results when worked out correctly.

Cauchy gave three ways of arriving at the equations, of which two set out from a molecular hypothesis similar to, but not identical with, that of Navier; viz. it is assumed that the solid

[1] *Mém. Acad. Sciences*, VII. Paris, 1827. The memoir was read in 1822.

[2] See Todhunter and Pearson, vol. I. arts. 266, 436, 443.

consists of a very large number of *material points*, with a law of force between pairs some function of their distance. In the first[1] of these "molecular" memoirs an expression is formed for the forces that act upon a single 'molecule', and the differential equations deduced; in the case of isotropy these contain two constants. In the second[2] expressions are formed for the stresses across any plane drawn in the solid. If the initial state be one of zero stress, and the solid isotropic, the stress will be expressed in terms of the strain by means of a single constant, and one of the constants of the preceding memoir must vanish. The equations are then identical with those of Navier, but they are obtained without replacing summations by integrations. In like manner, in the general case of æolotropy, Cauchy finds 21 independent constants, of which 6 vanish identically if the initial state be one of zero stress. These points were not fully explained by Cauchy. Clausius[3], however, has shewn that this is the meaning of his work. Clausius criticises the considerations of symmetry in molecular arrangement, by which Cauchy reduced his 15 constants to one in the case of isotropy, but the reduction can be effected by other methods, and the equations must be regarded as proved if the "molecular" hypothesis be admitted.

The first memoir by Poisson[4] relating to the same subject was read before the Paris Academy on April 14th, 1828. The memoir is very remarkable for its numerous applications of the general theory to special problems, but the treatment of the general equations is inferior to Cauchy's. Like Cauchy, Poisson first obtains the equations of equilibrium in terms of stress-components, and then estimates the stress across any plane resulting from the intermolecular forces. The expressions for the stresses in terms of the strains involve summations with respect to all the molecules, situated within the region of molecular activity of a given one. Poisson rightly decides against replacing the summations by integrations, but he assumes that this can be done

[1] 'Sur l'équilibre et le mouvement d'un système de points matériels'. *Exercices de Mathématiques*, 1828.

[2] 'De la pression ou tension dans un système de points matériels', same volume.

[3] 'Ueber die Veränderungen, welche in den bisher gebräuchlichen Formeln für das Gleichgewicht und die Bewegung elastischer fester Körper durch neuere Beobachtungen nothwendig geworden sind'. Pogg. *Ann.* 76, 1849.

[4] 'Mémoire sur l'équilibre et le mouvement des corps élastiques'. *Mém. Paris Acad.* VIII. 1829.

for the summations with respect to angular space about the given molecule, but not for the summations with respect to distance from this molecule. The equations thus obtained are identical with Navier's. The principle, on which summations may be replaced by integrals, has been explained as follows by Cauchy[1]:—If the molecular distribution be such that the number of molecules in any volume, which contains a very large number of molecules, and whose dimensions are at the same time small compared with the radius of the sphere of sensible molecular activity, be proportional to the volume, then, making abstraction of the molecules in the immediate neighbourhood of the one considered, the actions of all the others, contained in one of the small volumes referred to, will be equivalent to a force through the centroid of this volume, which will be proportional to the volume and to a function of the distance of the particular molecule from the centroid of the volume. The action of the remoter molecules is said to be "regular", and the action of the nearer ones, "irregular"; and thus Poisson assumed that the irregular action of the nearer molecules may be neglected, in comparison with the action of the remoter ones, which is regular. This is Sir G. Stokes's[2] description of Poisson's assumption, and it is the text upon which he founds his criticism of Poisson. Without making this assumption Cauchy arrived at Poisson's results.

Among later investigations of the stress-strain relations, as given by the molecular hypothesis, we must note especially those of Saint-Venant[3]. In the first place he gave an ingenious proof that, if the elastic reactions arise from changes in the molecular configuration, and the intermolecular forces are functions of the intermolecular distances, then, for very small strains, the stresses must be linear functions of the strains. For in this case the term of any stress, that arises from the force between two molecules, is the difference of the amounts of this force in the strained and unstrained states; and, since the force is supposed a continuous function of the molecular configuration, this difference must be ultimately a linear function of the variations in the intermolecular

[1] In his memoir first quoted. *Exercices de Mathématiques*, 1828, pp. 241—243 of the new edition. Paris, 1890.

[2] *Math. and Phys. Papers*, vol. I. pp. 116 sq. and *Camb. Phil. Soc. Trans.* VIII. 1845.

[3] See his edition of Moigno's *Statique*, and of Navier's *Leçons*, also the memoir on 'Torsion' and the 'Annotated Clebsch'.

distances; but these variations are linear in the strain-components —whence the result. Saint-Venant has also given a new proof that the constants in the expressions of the six stresses, in terms of the six strains, reduce to 15 in the most general case, provided the force between two molecules is in the line joining them and is a function of their distance. This proof does not depend on the formation of expressions such as Cauchy's for the constants in terms of the molecular configuration, but on a consideration of the changes of the molecular distances involved in the existence of the several strain-components.

Various attempts have been made to simplify or to get rid altogether of Navier's molecular hypothesis. The first of these is due to Cauchy[1]. He had proved the theorems (1) that the stress at any point can be expressed by means of three principal stresses, on three planes at right angles to each other, and normal to the planes across which they act, and (2) that the strain at any point can be reduced to three principal extensions, of three mutually perpendicular line-elements. He made assumptions, which amount to supposing (1) that the principal stresses are linear functions of the principal extensions, and (2) that, in an isotropic solid, the principal planes of stress are normal to the principal axes of extension. These assumptions lead to the equations of equilibrium of an isotropic solid, with two constants, in the form in which they are now generally accepted. Of these assumptions the first is a very special case of the generalised Hooke's law, and must rest on an experimental basis, but it was formulated by Cauchy without reference to experiment. The second seems to me to be much the most axiomatic of all the *assumptions* that have been proposed, and it is difficult to reconcile any contradiction of it with the notion of complete isotropy. Another theory of a similar character has been given much later by Maxwell[2], who proposed to assume (1) that the sum of the principal stresses is proportional to the sum of the principal extensions, (2) that the difference of any two principal stresses is proportional to the difference of the two corresponding principal extensions. The equations obtained by this method are the same as Cauchy's just referred to.

[1] 'Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide'. *Exercices*, 1828. This memoir precedes those in which the same author made use of the molecular method.

[2] 'On the Equilibrium of Elastic Solids'. *Edinburgh R. S. Trans.* xx. 1853.

Lamé, in forming the general equations, partly adopts and partly rejects the molecular hypothesis. In the joint memoir of this writer and Clapeyron of date 1828, Navier's method, with all its mistakes, was closely followed, though not attributed to its author. In his treatise on elasticity, however, Lamé[1] only invokes the molecular hypothesis to shew that the six components of stress are linear functions of the six components of strain, and the reduction of the constants to two, for isotropic solids, depends on considerations of symmetry. Although this treatise is of much later date than Green's investigation, to be presently noticed, Lamé seems to have been unacquainted with the method of the latter, and his work is more closely associated with the older school of Navier and Poisson than with the new school of Green and his followers.

The revolution which Green effected in the elements of elastic theory is comparable in importance with that produced by Navier's discovery of the general equations. Starting from what is now called the *Principle of the Conservation of Energy* he propounded a new method[2] of obtaining these equations. He himself stated his principle and method in the following words:—

"In whatever way the elements of any material system may "act upon each other, if all the internal forces exerted be multi- "plied by the elements of their respective directions, the total sum "for any assigned portion of the mass will always be the exact "differential of some function. But this function being known, "we can immediately apply the general method given in the "*Mécanique Analytique*, and which appears to be more especially "applicable to problems that relate to the motions of systems "composed of an immense number of particles mutually acting "upon each other. One of the advantages of this method, of "great importance, is that we are necessarily led by the mere "process of the calculation, and with little care on our part, to all "the equations and conditions which are *requisite* and *sufficient* for "the complete solution of any problem to which it may be applied."

The function here spoken of, with its sign changed, is the potential energy of the strained elastic solid per unit of mass, expressed in terms of the components of strain, and, if the function

[1] *Leçons sur la théorie mathématique de l'élasticité des corps solides*, 1852.

[2] *Camb. Phil. Soc. Trans.* VII. and *Math. and Phys. Papers*. The date of Green's memoir is 1837.

exist, its differential coefficients, with respect to the components of strain, are the components of stress. Sir W. Thomson has shewn[1] that the function does not in general exist, unless, either the solid is strained at constant temperature, or the strain is effected so quickly that no heat is gained or lost by any element of the solid, and that in these cases its existence is a consequence of the second law of Thermodynamics, and not, as Green supposed, of the principle of the conservation of energy.

Green supposed his function expressible in terms of the components of strain, and capable of being expanded in powers and products of these components. He therefore arranged it as a sum of homogeneous functions of the strain-components of the first, second, and higher degrees. Of these, it can be shewn that the first disappears, as the potential energy must be a true minimum when the solid is unstrained, and, as the strains are all small, the second term will alone be of importance. From this principle Green deduced the equations of elasticity, containing in the general case 21 constants, which reduce to two in the case of isotropy.

The method thus introduced by Green has been followed by most English and German mathematicians, and has been much developed by Kirchhoff[2] and Sir W. Thomson. It has received severe criticism at the hands of Saint-Venant. Before proceeding to its discussion, it will be best to notice the theories propounded by Clebsch and Sir G. Stokes.

The latter[3] was the first to observe that the generalised Hooke's law, of the proportionality of stress and strain, is a consequence of the experimental fact that all solids admit of being thrown into a state of isochronous vibration. It follows from this law, and from considerations of symmetry, that in an isotropic solid a uniform dilatation is opposed by a hydrostatic pressure proportional to the dilatation, and that a uniform shear of any plane is opposed by a shearing (tangential) stress in that plane, proportional to the shear. From these observations, the equations of elasticity were deduced, involving two constants. Sir G. Stokes's memoir is remarkable for the continuity it attempts to

[1] *Quarterly Journal*, v. 1857.

[2] *Vorlesungen über mathematische Physik, Mechanik.*

[3] 'On the theories of the...Equilibrium and Motion of Elastic Solids'. *Camb. Phil. Soc. Trans.* VIII. 1845.

trace from perfect fluids to perfectly elastic solids through plastic solids, and his defence of the equations with two constants depends partly on this supposed continuity of behaviour of materials of different structure.

Clebsch's theory[1] is interesting on account of his dispensing with any physical hypotheses or experimental data whatsoever. He says in effect that, as the strains are all small, and the stresses are functions of the strains, which vanish when these vanish, the stresses can be expanded in homogeneous functions of the strains, of which only the terms of degree unity need be retained. Saint-Venant has pointed out that, even if we might assume that the stresses can be expanded in integral powers of the strains, which is not necessarily true *a priori*, we should have no right to predict that the first powers occur in the expansion, and he remarked that the stress-strain relation is a matter to be determined by experiment, except in so far as it can be deduced from a knowledge of the intermolecular action between the parts of the solid.

We have had frequent occasion to notice a discrepancy in the number of elastic constants which are found in the equations obtained from different theories. In case these equations are deduced from a molecular hypothesis such as Navier's, they involve fewer constants than when they are derived by methods like those of Green and Sir G. Stokes, and it is a very important question whether the relations among the constants in Green's equations, necessary to reduce them to Navier's equations, really hold. The questions to be discussed are whether elastic æolotropy is to be characterised by 21 constants or 15, and whether elastic isotropy is to be characterised by two constants or one. The two theories are styled by Prof. Pearson[2] the multi-constant theory and the rari-constant theory respectively. Among rari-constant elasticians the most prominent are Navier, Poisson and Saint-Venant, while in the writings of Cauchy and Lamé sometimes one theory is adopted and sometimes the other. Green, without intending it, is practically the founder of the multi-constant theory, though it had been introduced by Cauchy in his first memoir on the general equations. In Lamé's treatise we have multi-constant equations deduced from an hypothesis which ought to have led him to rari-constancy. Sir G. Stokes was the first to insist on

[1] *Theorie der Elasticität fester Körper*, 1864.

[2] Todhunter and Pearson, vol. I. arts. 921 sq.

the importance of the discrepancy, and Sir W. Thomson has been the most strenuous opponent of the rari-constant theory.

This theory rests on the hypothesis that the action between two molecules is in the line joining them, and is a function of their distance. In other words it proceeds on the assumption that the behaviour of solid bodies is the same as it would be if they were composed of an immense number of *material points*, between which are forces of attraction or repulsion, following a certain law. The working out of this hypothesis leads to certain relations among the constants, by which the six components of stress, at any point of a solid, are expressed in terms of the six components of strain, which relations ought to admit of experimental verification. I call these relations *Cauchy's relations*, because they are virtually included in his investigation, although they appear to have been first formulated by Saint-Venant in his great memoir on the torsion of prisms. The particular case of isotropy is the best known. For this case, Navier, Poisson, and Cauchy deduced from the molecular hypothesis equations containing a single elastic constant, while the utmost reduction, that can be effected without recourse to this hypothesis, leaves two independent constants. The result which ought most easily to admit of experimental verification is concerned with the ratio of the linear lateral contraction to the linear longitudinal extension of a bar under terminal tractive load. According to the rari-constant theory this ratio must be $\frac{1}{4}$ for all isotropic materials. According to the multi-constant theory it depends on the material and may vary between the extreme values $\frac{1}{2}$ and -1[1]. The supporters of the rari-constant theory rely[2] partly on the experimental evidence, which they hold to be definitely favourable to their view, and partly on the value of the hypothesis from which it is deduced. They urge, in favour of this hypothesis, the general consent that has been accorded to it since it was first propounded by Newton, its success in explaining the phenomena of gravitation and the conservation of energy, and the similar success of similar hypotheses in the kinetic theory of gases, and in the theories of electricity and magnetism. The opponents of the theory urge against it firstly that it rests on a hypothesis possibly doubtful,

[1] See below, ch. III., art. 28.

[2] See in particular Saint-Venant's edition of Navier's *Leçons sur l'application de la Mécanique*, where the subject is discussed at length in *Appendice* V.

secondly that this hypothesis has been incorrectly worked out, thirdly that it contradicts the results of experiment, and lastly that the known laws of energy lead to results which are certainly true, whether the molecular hypothesis be correct or no, and these laws are sufficient to serve as a basis for theory.

Of these objections the first depends entirely on our view of the world. The older theories of physics were content with explaining phenomena by the assumption of elements acting upon each other at a distance. A dynamical explanation of any phenomenon once consisted in a statement of the attracting and repelling forces adequate to produce it. Why these forces existed, how they arose, were questions on which science was dumb. Modern speculations in molecular dynamics point in the direction of a kinetic theory of matter, according to which all the interactions between portions of matter are effected through the intervention of a continuous medium. If we are to obtain equations of elasticity from a supposition of this kind, without knowing the nature of the medium and the nature of the atoms, we can only invoke the known laws of energy, as was done by Green and Sir W. Thomson, but we are not thereby placed in a position to prove that the molecular hypothesis in question is not an adequate mathematical representation of the facts.

I do not think it can be successfully contended that the hypothesis could properly lead to any but rari-constant equations. It is true that errors occur in the earlier writings on the subject, which have been seized upon by the opponents of the theory, and held to invalidate its results. There is in fact no single investigation which would be entirely acceptable to modern mathematicians, but the explanations which Clausius has given of one of Cauchy's memoirs prove that that memoir might have been so written as to shew that the hypothesis really leads to Poisson's equations, although not strictly by Poisson's method of investigation. Sir W. Thomson[1] has indeed endeavoured to prove that the theory is self-contradictory. This he proposed to do by actually constructing a model of a molecule, which shall possess æolotropy of the most general kind supposed by Green, all the parts of the model being made of isotropic material fulfilling Poisson's condition. I fail to see how an unbiassed judge could accept the model as the proof of

[1] *Lectures on Molecular Dynamics.* Johns Hopkins University, Baltimore, 1884.

a flaw in Cauchy's analysis. More recently Sir W. Thomson[1] has convinced himself that there exists a law of intermolecular force, between "Boscovich point-atoms", which would lead to rari-constant equations for an isotropic solid, whose elements are such atoms, so that perhaps we may regard the contention of incorrect working out as given up.

Both sides in this controversy appeal with equal confidence to the confirmation of their views by experimental investigations. It would seem, at first sight, a simple matter to determine, in some form, two moduluses of elasticity of a great number of isotropic substances, and to observe whether their ratio is that which it should be on the rari-constant hypothesis. The contention of the rari-constant elasticians is that the result confirms their view, whenever reasonable care has been taken to perform the experiments upon an isotropic elastic solid, strained within its limits of elasticity. They reject, as worthless, experiments on such solids as cork and india-rubber, which contain numerous cavities of dimensions incomparably greater than those of the sphere of molecular activity. They explain many apparent contradictions of their theory, offered by experiments on wires, by the supposition that the solid subjected to experiment was really æolotropic; and although there are æolotropic materials whose elastic properties are expressed on the rari-constant theory by two constants, the formulæ for these are quite different from those of biconstant isotropy. Until very recently their opponents relied generally on experiments made on wires or thin plates probably very æolotropic, but treated as isotropic, or else on the continuity first suggested by Sir G. Stokes in the behaviour of different kinds of materials, ranging from perfect fluids to perfectly elastic solids, and including such solids as cork, jelly, and india-rubber among elastic solids. The continuity referred to consists really in the continuously changing relative importance of set and elastic strain, in different classes of materials—an appeal is virtually made to experiments on something else than elastic solids to disprove a supposed property of the latter. It seems to me unfortunate that the supporters of multi-constancy should have taken up this line of argument, as, at any rate since 1860, exact methods of experimental investi-

[1] 'Molecular constitution of Matter.' *Edinburgh R. S. Proc.* 1889. See also *Math. and Phys. Papers*, vol. III.

gation have been within their reach. Numerous researches have in fact been made for the express purpose of discovering the true value of Poisson's ratio for various solids. Among them it is true that some were not conducted with proper care, but it is not a little remarkable that they all agree in finding values of this ratio which differ for different materials, and occasionally they find the ratio almost exactly equal to $\frac{1}{4}$. The first of such researches is that of Wertheim[1] who was led to take up the subject by an experiment of Cagniard Latour's on compression, by which a result was obtained that appeared to be in conflict with one found analytically by Lamé. The materials selected by Wertheim were glass and brass, and he found that, for both, the ratio is nearer to $\frac{1}{3}$ than $\frac{1}{4}$. No great importance can be attached to these experiments as the material was probably not isotropic, but later experimenters have taken more care. Kirchhoff[2] devised experiments on the torsion and flexure of steel bars, using Saint-Venant's formulæ. These experiments yield a direct comparison of moduluses, and consequently the value of Poisson's ratio, which he found to be ·294 for his materials. More recently experiments by M. Amagat[3] on the compressibility of solids, conducted with great care, led him to values of Poisson's ratio which vary from about $\frac{1}{4}$ for glass to ·428 for lead, and verify Wertheim's value for brass, these experiments, like those of Kirchhoff, were made with full knowledge of the nature of the point in dispute. But perhaps the most striking experimental evidence is that which Prof. Voigt[4] has derived from his study of the elasticity of crystals. The objection to materials possibly æolotropic, but treated as isotropic, was got rid of when he had the courage to undertake experiments on materials known to be æolotropic in a given manner[5]. The point to be settled is however more remote. According to Green there exist, for a solid of the most generally æolotropic character, 21 independent elastic constants. The molecular hypothesis, as

[1] *Annales de Chimie et de Physique*, XXIII. 1848.

[2] Pogg. *Ann.* CVIII. 1859.

[3] *Journal de Physique*, VIII. 1889.

[4] Wiedemann's *Annalen*, XXXI. 1887, XXXIV. and XXXV. 1888, and XXXVIII. 1889.

[5] It may be questioned whether this can be *known* in the manner assumed by Prof. Voigt following in the footsteps of F. E. Neumann. See ch. III. of the present work.

worked out by Cauchy and supported by Saint-Venant, leads to only 15 constants, so that, if the rari-constant theory be correct, there must be 6 independent relations among Green's 21 coefficients. These relations I call Cauchy's relations. Now Prof. Voigt's experiments were made on the torsion and flexure of prisms of various crystals, for most of which Saint-Venant's formulæ for æolotropic rods hold good, for the others he supplied the required formulæ. In the cases of Beryl and Rocksalt only were Cauchy's relations even approximately verified, in the seven other kinds of crystals examined there were very considerable differences between the coefficients which these relations would require to be equal. The most remarkable results of this kind are those for the regular crystal Pyrites, for which the two coefficients that ought to be equal are respectively -483×10^6 and 1075×10^6 grammes' weight per square centimetre. The latter is the principal rigidity, or resistance to shear of planes perpendicular to an axis of the crystal, and is considerably greater than the rigidity of steel, the former is negative and large, being comparable with this rigidity. Exactly similar results were obtained from numerous experiments on rods of the material. It appears to me that a single result of this kind, once firmly established, is sufficient to discredit the molecular or rather *point-atom* hypothesis as a basis for elastic theory.

Even if the experimental evidence were all fairly interpretable in favour of the other side, if there were a general consensus that Cauchy's relations hold good, and that Poisson's ratio is $\frac{1}{4}$, for materials carefully examined, that would not amount to a proof of the molecular hypothesis. It would still be open to us to reject that hypothesis as *not axiomatic*, and in the present state of science we must so reject it. It is futile to argue, as Saint-Venant does, that, because some proofs of the principle of energy rest on the assumption of central intermolecular force, therefore a system of forces, even if it have a potential, cannot be conservative unless the force between two molecules is central, and a function of their distance. Unless the hypothesis were axiomatic, there could be no reason to adopt it to-day. Modern Physics is perfectly capable of deducing a theory of elasticity from the known laws of energy, without the aid of a subsidiary hypothesis about intermolecular force, and, being in that position, it is bound to discard the hypothesis. Such a device is merely a phase in the develop-

ment of scientific thought, and, having served its turn as a means of introducing generality into the subject, it must give place again to a still more general method.

Saint-Venant as the champion of rari-constancy has repeatedly urged an objection against Green's method which its supporters do not appear to have met directly. The step objected to is the supposed possibility of expanding the energy-function in terms of the strain-components, and the retention of the second term. It is true that, as a proposition in pure mathematics, the step is unjustifiable. We have no right to assume that because one quantity depends upon another, and the first vanishes, and has a minimum value when the second vanishes, that, therefore, the first can be expanded in powers of the second, and terms of the second order occur. Many examples could be given to the contrary. But it is different in the case of elasticity. There is a definite physical reason, not stated by Green, and not generally stated in that connexion by his followers, viz.:—that experiment shews that the stress, in an elastic solid strained at constant temperature, or executing small vibrations, is a linear function of the strain, and it follows from this, analytically, that the potential energy of strain if a function of the strains at all, is a quadratic function of the strains, when the latter are small. That the potential energy is a function of the strains in these two cases is a proposition in Thermodynamics, first proved by Sir W. Thomson.

We have just seen that the modern theory of elasticity rests upon the generalised Hooke's Law, as a fundamental datum given in experience. It is therefore necessary to pay some attention to the history of science in respect of this law. Its discovery by Hooke and Marriotte has already been noticed, but the experiments which led them to it were not of a very conclusive character. James Bernoulli, the discoverer of the elastic line, challenged it in 1744. The mathematicians of the 18th century assumed the linearity of the relation between tension and extension, whenever they needed it. For this case, Young gave precision to the law by the introduction of his modulus. Hodgkinson's experiments on cast-iron led him to conclude that, for this material at any rate, the law does not hold good. The discoverers of the general equations of elasticity, Navier, Poisson, and Cauchy, could all have deduced it from their molecular hypothesis if they had paid

attention to the point, but they did not. This was reserved for Saint-Venant and Lamé. The point was really settled in 1845, when Sir G. Stokes remarked that the capacity of all solids to execute isochronous vibrations proves that the stress-strain relations must be linear for the very small displacements involved. It is sufficient for the mathematical theory as at present developed to know that the law is true for infinitesimal strains. It is a matter of interest, for possible future developments, to know further that, for all solids, (except cast-iron and perhaps some other cast metals), the law represents the stress-strain relation, as accurately as experiment can tell, for finite strains within the elastic limits.

Now just as the generalised Hooke's Law was introduced into the mathematical theory from the analytical rather than the physical side, so almost the whole machinery of coefficients of elasticity, expressing the law, comes from the same source. Young's modulus, as a coefficient, is practically in the old theories of beams, in vogue before the time of Young. The rigidity, or coefficient of resistance to shearing strain, was in mathematical memoirs, (of course without a name), before it was suggested by Vicat[1] and defined by Navier[2]. The whole set of 21 coefficients of Green's energy-function remained unnamed till the appearance of Rankine's paper of 1855[3]. But, after the introduction of A's and B's to express properties of matter, the physicist has come forward with an explanation as to what property of matter is expressed by A or B, his work has been a nomenclature of the A's and B's depending on something concrete which they really express, or the discovery of relations between the coefficients and some possible new set expressing simpler properties.

In the theory of isotropic solids there occur two constants at most, say the K and k of Cauchy's first memoir. If Poisson's ratio be $\frac{1}{4}$, $k = 2K$. Cauchy's equations involving these constants are obtained by means of rather arbitrary assumptions. Different writers use different constants, which can be expressed in terms of Cauchy's. Navier and Poisson use a single constant, and so in other writings

[1] 'Recherches expérimentales sur...la rupture'. *Annales des ponts et chaussées*, Mémoires 1833.

[2] In the second edition of his *Leçons*, 1833.

[3] 'On Axes of Elasticity and Crystalline Forms'. Rankine's *Miscellaneous Scientific Papers*.

does Cauchy. Lamé and Clapeyron use a single constant. Meanwhile Young's modulus is already defined physically. Presently comes Vicat with a physical definition of the rigidity. What is the relation of these physical constants to the coefficients in the elastic equations? There is no answer, but Green appears instead with two new constants A and B which he shews depend on the velocities of plane waves in the solid. Sir G. Stokes follows with again two new constants, defined, this time, from physical considerations. One is Vicat's rigidity, the other is the modulus of compression, or the ratio of a hydrostatic pressure applied uniformly to a solid to the cubical compression it produces. Then comes Lamé with his constants μ and λ, obtained rather in the manner of Cauchy's K and k, easily expressible in terms of those of Sir G. Stokes or Green, of whose writings he appears ignorant, μ is in fact the rigidity. Kirchhoff follows with his K and θ, of which K is the rigidity and θ a number, these are introduced like Green's A and B as coefficients in the energy-function. In reading any memoir it is necessary to have some acquaintance with six constants, the more or less arbitrary pair used by the writer of the memoir, the modulus of compression, the rigidity, Young's modulus, and Poisson's ratio.

For æolotropic solids the matter is much simplified by the comparative smallness of the literature. Green introduced his 21 coefficients, and gave little explanation of them. Franz Neumann[1] was the first to use the coefficients of Green's energy-function to express the elastic properties of crystals. He assumed that crystallographic symmetry corresponds to symmetry in elastic quality, and he thence shewed how to find the proper reductions in the number of the constants for the holohedral forms of the six classes of crystals, and, for systems having three planes of symmetry, he further shewed how to express the Young's modulus of the material, in a given direction, in terms of the coefficients. This theory has received much attention at the hands of Saint-Venant[2]. Prof. Voigt[3] has extended Neumann's work so as to include the principal hemihedral crystalline forms,

[1] *Vorlesungen über die Theorie der Elasticität der festen Körper und des Lichtäthers*, 1885. The lectures were delivered in 1857—8.

[2] 'Mémoire sur la distribution des élasticités autour de chaque point'. Liouville's *Journal de Mathématiques*, VIII. 1863.

[3] Wiedemann's *Annalen*, XVI. 1882.

and has developed the theories of flexure and torsion, so as to obtain experimental methods for determining the constants of crystals with high degrees of symmetry. We have already seen how his experiments throw light on the *constant* controversy. The most important of Saint-Venant's researches, in this part of the subject, relates to the formula, which gives Young's modulus for any direction in an æolotropic solid with three planes of symmetry. Neumann had shewn that the modulus in any direction is proportional to the inverse fourth power of the radius-vector of a certain quartic surface, the coefficients in which are functions of the coefficients of elasticity[1]. Saint-Venant proved that this radius-vector has 13 maxima and minima, but, if certain inequalities among the elastic coefficients be fulfilled, all but three are imaginary. It appears not unlikely that the maxima and minima of the Young's modulus should belong to principal axes of symmetry only. Saint-Venant also investigated the values of Poisson's ratio for extension in the direction of one axis, and contraction in that of another. He applied these researches to obtain formulæ that might prove useful in the case of timber and laminated metals, which have a certain æolotropic character without being crystalline. Another matter, to which he drew attention[2], was the possibility of the directions of the principal axes of symmetry of contexture of a material, varying, from point to point, according to a definite law, so that, when suitable curvilinear coordinates are employed the stresses may be expressed in terms of the strains by formulæ which hold for all points, and he applied this theory to obtain results suitable for the explanation of certain piezometer experiments by Regnault, in which a shell of metal, forming part of the apparatus, probably has such a kind of æolotropy.

Two other points should be noticed in connexion with the elastic constants. One is that they vary with the temperature. In general a rise of temperature is accompanied by a decrease in the values of the constants. This point has been established chiefly by the experiments of Wertheim[3], Kohlrausch[4] and Mr

[1] See Saint-Venant's 'Annotated Clebsch'. *Note du* § 16.

[2] 'Sur les divers genres d'homogénéité des corps solides'. Liouville's *Journal*, 1865.

[3] 'Recherches sur l'élasticité'. *Annales de chimie*, XII. 1844.

[4] Pogg. *Ann.* CXLI. 1870.

Donald McFarlane[1]. The other is that the constants in the equations of vibration are not identical with those in the equations of equilibrium. This may be illustrated by a reference to Laplace's celebrated correction of the Newtonian velocity of sound. In the case of vibrations, the changes of state follow the adiabatic law, no heat being gained or lost by any element; in the case of strain gradually produced at constant temperature, the changes of state, following the isothermal law, differ from those that have place in a vibrating solid. The moduluses in the two cases are called by Sir W. Thomson kinetic and static moduluses respectively, and the latter are a little smaller than the former, but the ratio is very much nearer to unity for solids than for air. This point seems to have been first investigated by Lagerhjelm[2] in 1827.

Before passing to the consideration of problems, it is proper to notice some other matters connected with the general theory. These are the thermo-elastic equations of Neumann and Duhamel, the transformation of the equations of elasticity to orthogonal curvilinear coordinates, the theory of the propagation of disturbances by wave-motion in an unlimited elastic solid medium, and the general theory of the free vibrations of solids.

One method by which the ordinary equations of elasticity have been obtained is, as we have seen, to assume that an elastic solid behaves like a system of material points, between which are forces of attraction or repulsion, and to estimate the stress thence arising when alterations are made in the intermolecular distances. When the temperature is variable, the force cannot be taken simply a function of the distance. Duhamel[3] assumed that there is in this case an additional term in the force, proportional to the increase of temperature, and he thence obtained equations for the equilibrium of a solid strained by unequal heating. Franz Neumann[4] about the same time obtained similar equations by a method, which amounted to assuming that in a small part of a solid, so strained, there is a uniform elastic pressure proportional to the

[1] Quoted by Sir W. Thomson, art. *Elasticity*, *Encyc. Brit.* and *Math. and Phys. Papers*, vol. III.

[2] See Todhunter and Pearson, vol. I. art. 370.

[3] 'Mémoire sur le calcul des actions moléculaires développées par les changements de température dans les corps solides'. *Mém....par divers savans*, v. 1838.

[4] 'Die thermischen...Axen des Krystallsystems des Gypses', Pogg. *Ann.* XXVII. 1833, and 'Die Gesetze der Doppelbrechung...', *Abh. k. Akad. Wiss.* Berlin, 1841; see also the same author's *Vorlesungen über die Theorie der Elasticität...*

temperature. The thermodynamical investigation of Sir W. Thomson shews that these equations cannot be deduced from known laws, and experiment appears to shew that the temperature coefficient introduced by Neumann and Duhamel is not constant but a function of the strains. We must regard the thermo-elastic equations of these writers as a provisional suggestion, destined to give place to a theory founded on fuller experimental knowledge.

To Lamé belongs the credit of introducing the methods of curvilinear coordinates into the study of physics. In a sense the whole theory is due to him. Special cases had received treatment before his time, but we owe to him all the fundamental general theorems of the subject. He succeeded in transforming the equations of elasticity to orthogonal coordinates, and gave, in his *Leçons sur les Coordonnées Curvilignes*, the values of the strain-components, and the equations for the stresses. He also gave the equations determining the displacements when the solid is isotropic. In elastic theory the most important cases are those of spherical and cylindrical coordinates. These have been treated by Mr Webb[1] by means of vector-differentiation depending on the kinematical method of "moving axes" introduced by Mr R. B. Hayward[2]. Other investigations have been given by Saint-Venant and Borchardt[3], and Mr Larmor[4] has shewn how to deduce the equations from a knowledge of the formula for the line-element, and the energy-function.

The theory of the propagation of waves in an unlimited isotropic elastic medium was first considered by Poisson, who, in his memoir of 1828, shewed that there are two kinds of waves, one waves of compression, and the other waves of distortion, and that these are propagated independently with different velocities. He also gave the now well-known integral of the equations of wave-propagation, which expresses the motion, at any place and time, in terms of the initial disturbance. The interpretation of this integral was given much later by Sir G. Stokes[5]. Green con-

[1] *Messenger of Mathematics*, 1882.

[2] *Camb. Phil. Soc. Trans.* VII. 1856.

[3] *Crelle-Borchardt*, LXXVI. 1873.

[4] *Camb. Phil. Soc. Trans.* XIV. 1885.

[5] 'On the Dynamical Theory of Diffraction'. *Camb. Phil. Soc. Trans.* IX. 1849, and *Math. and Phys. Papers*, vol. II.

sidered the propagation of plane waves in an æolotropic medium[1], and concluded that there are three kinds of waves which are propagated with different velocities. When the medium is isotropic the cubic equation giving these velocities has two equal roots. The theory of wave-motion in an æolotropic medium was given by Blanchet in two memoirs in Liouville's *Journal* (v. 1840, and vii. 1842). He integrated the equations of wave-propagation, and interpreted his integrals so as to lead to the wave-surface method of physical optics. A different investigation has been given by Herr Christoffel[2]. All the developments of this theory belong to physical optics. Optical phenomena lead to the hypothesis of a medium in which waves of light can be propagated, and it is a definite question whether the properties to be attributed to the medium, in order that the results may be in accordance with observation, are identical with those of an elastic solid; and it is therefore very important, for optical theory, to have an account of the propagation of a disturbance in an elastic solid medium. Thanks to the investigators referred to, we have such an account.

The general theory of the free vibrations of solids is due to Clebsch, and appeared for the first time in his treatise of 1864. Particular problems had previously been treated by Euler, Poisson, Kirchhoff, and others, but Clebsch appears to have been the first to formulate true general results which apply to all solids. To him must be attributed the extension of the notion of principal oscillations to systems with an infinite number of degrees of freedom, and the introduction of the corresponding normal functions, with the proof of their principal properties; he also pointed out the utility of the variational equation of motion in investigating these properties, and that of the boundary-conditions in determining periods and types. This theory was given by Clebsch as a generalisation of Poisson's theory of the radial vibrations of a sphere (published in 1828), but it was no doubt also in part suggested by the already well-known results for strings, bars, plates, and membranes. Lord Rayleigh[3] went further, in connecting the theory with the purely dynamical

[1] 'On the propagation of light in crystallized media.' *Camb. Phil. Soc. Trans.* vii. 1842.

[2] 'Fortpflanzung des Stosses...' Brioschi's *Annali di Matematica*, viii. 1877.

[3] *Proc. Lond. Math. Soc.* iv. 1873, and *Theory of Sound*, vol. i. 1877.

treatment of small oscillations about a configuration of stable equilibrium, and extending to it some new theorems relating to such oscillations in a system with finite freedom.

Before the appearance of Clebsch's treatise, a different theory had been propounded by Lamé in his *Leçons sur...l'élasticité.* Acquainted with Poisson's discovery of waves of compression and waves of distortion, he concluded that the vibrations of any solid must fall into similar classes, and he investigated the vibrations of various bodies on this assumption. The fact that his solutions do not satisfy the boundary-conditions that hold at the surfaces of his solids, is a sufficient disproof of his theory; but it was finally disposed of when Prof. Lamb shewed how to calculate all the modes of vibration of a homogeneous isotropic sphere, proving that the classes, into which they fall, do not verify Lamé's supposition.

The general problems of the theory of elasticity may be stated as follows:—

(1) A body of any form is subject to the action of any given bodily forces, and surface-tractions, or has its surface deformed in any given manner, it is required to determine the state of strain and displacement in the interior.

(2) The body executes small vibrations, either freely, or under the action of given periodic forces, it is required to find the modes and periods of the small free oscillations, and the amplitude of the forced oscillations.

We have now to consider what degree of success has attended the efforts of mathematicians to solve these problems.

The first general solution was given by Lamé and Clapeyron in their memoir of 1828, where there is an investigation of the displacement produced in the interior of an isotropic solid bounded by an infinite plane, at whose surface there is a given distribution of load. This problem of the infinite plane has been the subject of researches by several writers. When the load is a harmonic distribution of normal pressure it is not difficult to find a solution by means of Fourier's series, such solutions have been considered by several writers[1]. Another method has been followed by M. Boussinesq[2]. Lamé[3] had noticed that certain potential functions

[1] See e.g. *Solutions of the Cambridge Problems...*, *for* 1875, pp. 150 sq.

[2] The researches of this author on this part of the subject commence with four papers in the *Comptes Rendus*, LXXVIII. 1879, and culminate in his *Applications des Potentiels....* Paris, 1885.

[3] *Leçons sur...l'élasticité, sixième leçon.*

could be applied to obtain solutions of the equations of elastic equilibrium; these are the "inverse potential", *i.e.* the ordinary potential, or volume-integral of the product of a given function and the reciprocal of the distance of any point within a certain region from a given point, and the "direct potential" or volume-integral containing the distance. M. Boussinesq added to these the "logarithmic potential" which is the similar volume-integral containing the logarithm of a certain function of the coordinates, and he gave, in terms of potential functions of certain surface-distributions, a solution of the problem of the equilibrium of a solid bounded by an infinite plane, on which there is an arbitrary distribution of normal pressure. The general problem was solved by Signor Valentino Cerruti[1] who applied to it a general method of integrating the equations of elastic equilibrium, devised by Prof. Betti. The displacement at any point is expressed in terms of surface-integrals, involving the arbitrary distribution of surface-displacement or surface-traction. M. Boussinesq afterwards developed his theory of potential functions, so as to obtain the solutions of Signor Cerruti, and he considered particular cases in considerable detail. Of these, the most interesting is the case of a solid deformed by considerable pressure, applied in the neighbourhood of a single point of its surface; and the consideration of this case led to a remarkable theory of "local perturbations", according to which the effect of force, applied in the neighbourhood of any point of a body, falls off very rapidly as the distance from the point increases, and in particular the application of an equilibrating system of forces to a small part of a solid produces an effect, which is negligible at considerable distances from the part, so that in estimating the effect produced at a distance, by force applied in any manner near to a given point, the resultant only of the forces need be taken into account, their mode of application being comparatively insignificant. This is of importance in connexion with Saint-Venant's and many other problems.

After the plane, the next surface discussed was the sphere. This problem was first considered generally by Lamé, who gave a complete solution, in terms of spherical harmonics, of the case where an isotropic sphere, or spherical shell, is subject to its own

[1] 'Ricerche intorno all' equilibrio de' corpi elastici isotropi'. *Reale Accademia dei Lincei*, Rome, 1882.

gravitation, and to any distribution of surface-traction. Lamé[1] commenced by transforming the equations to polar coordinates. The equations of the problem in rectangular coordinates were first solved by Sir W. Thomson[2], who applied the results to the consideration of astronomical problems relating to the elastic equilibrium of the earth, deformed by tide-generating forces, or centrifugal force. In the case of the tides it was shewn that the degree of rigidity to be attributed to the solid, in order that ocean tides upon it may be similar to those on the earth, is very considerable, and the result discredits somewhat the geological hypothesis of the internal fluidity of the earth. The application of the problem to test this hypothesis is however beset with difficulties which have not yet been surmounted. The spherical harmonic solutions of the equations of elasticity have an extended application to other problems besides that of the equilibrium of the sphere. They are solutions in terms of integral powers of the coordinates, and they have been considered in this light by Mr Chree[3], who has shewn, by means of them, how to obtain a solution of some problems relating to the equilibrium of ellipsoids, and has also utilised them to verify Saint-Venant's solutions for the torsion and flexure of beams. Another application of them which has been recently made[4] is to investigate the effect of flaws in diminishing the strength of structures, verifying for the simplest case the factor of safety 2, allowed by engineers to guard against this form of weakness. A different solution of Lamé's problem has been given by Borchardt[5]. Instead of spherical harmonic series the displacements are expressed in terms of definite integrals involving the given surface-tractions, and a like solution has been given, by the same writer, of the problem of the strain in a sphere deformed by unequal heating[6], setting out from the thermo-elastic equations of

[1] Liouville's *Journal*, XIX. 1854.

[2] *Phil. Trans. R. S.* 1863.

[3] 'A new solution of the equations of an isotropic elastic solid, and its application to the theory of beams'. *Quarterly Journal*, 1886. See also another paper by the same author in the same journal, 1888.

[4] Larmor, *Phil. Mag.* Jan. 1892.

[5] 'Ueber Deformationen elastischer isotroper Körper durch mechanische an ihre Oberfläche wirkende Kräfte'. *Berlin Monatsberichte*, 1873.

[6] 'Untersuchungen über die Elasticität fester isotroper Körper in Berücksichtigung der Wärme'. *Berlin Monatsberichte*, 1873. This paper and the one last referred to are reprinted in Borchardt's *Gesammelte Werke*.

Neumann and Duhamel. The method of Lamé, consisting partly in the transformation to appropriate coordinates, has been applied by Herr Wangerin[1] to obtain solutions of the general equations, for a solid bounded by a surface of revolution, for which Laplace's equation can be solved.

The only general method that has been devised is that of Prof. Betti[2] mentioned above. He set out from a general reciprocal theorem, which can be stated in the form:—The whole work done by forces of any type, acting over the displacements produced by forces of a second type, is equal to the whole work done by the forces of the second type, acting over the displacements produced by those of the first. He shewed how to obtain the solution of the equations for any arbitrary distribution of surface-displacement, or surface-traction, in terms of the corresponding solution for certain particular distributions. The solution, that would be obtained by this method, puts in evidence the surface-displacement or surface-traction arbitrarily given, and is analogous to the solution of problems in electrostatics by means of Green's function. There can be little doubt that the method was suggested by electrical theory. Prof. Betti has applied it to the sphere-problem, and obtained results identical with those of Borchardt, and we have seen that, in the case of the plane-problem, success attended the application of it by Signor Cerruti.

Excepting the special problems of thin wires and plates, the problem of the vibrations of a given solid has been solved only in the case of the sphere and spherical shell. The radial vibrations of the sphere were first considered by Poisson in 1828 and served as the text on which Clebsch explained his theory of the free vibrations of solids. The analysis of the general problem was first completely given by Herr Jaerisch[3], who shewed that the solution could be expressed in terms of spherical harmonics and certain functions of the radius, which are practically Bessel's functions of order *integer* $+\frac{1}{2}$. This result was obtained independently by Prof. Lamb, who gave[4] an account of all the simpler modes of vibration, the nature of the nodal divisions of the sphere when any

[1] 'Ueber das Problem des Gleichgewichts elastischer Rotationskörper', Grunert's *Archiv*, LV. 1873.

[2] *Il Nuovo Cimento*, VI—X. 1872 sq.

[3] *Crelle-Borchardt*, LXXXVIII. 1879.

[4] *Proc. Lond. Math. Soc.* XIII. 1882.

normal vibration is executed, and the periods; we have already remarked upon the utility of this solution in regard to the general theory of the vibrations of solids.

Prior to the discovery of the general equations there existed theories of the torsion and flexure of beams starting from Galilei's enquiry and a suggestion of Coulomb's. The problems thus proposed are among the most important for practical applications, as most questions that have to be dealt with by engineers can, at any rate for the purpose of a rough first approximation, be reduced to questions of the resistance of beams. Cauchy was the first to attempt to apply the general equations to this class of problems, and his investigation of the torsion of a rectangular prism[1], though not correct, is historically important, as he recognised that the normal sections do not remain plane. His result had little influence on practice. The practical treatises of the earlier half of the present century contain a theory of torsion with a result that we have already attributed to Coulomb, viz.: that the resistance to torsion is the product of an elastic constant, the amount of the twist, and the moment of inertia of the cross-section. In Young's *Lectures on Natural Philosophy* and in Navier's *Leçons sur l'Application de la Mécanique* this is attributed to the relative displacement of the normal sections of a twisted prism, *i.e.* really to the shear, though this is not distinctly stated by Navier, and it is assumed that the normal sections remain plane. Again, in the theory of flexure, the practical treatises of the time followed the Bernoulli-Eulerian theory, attributing the resistance to flexure entirely to extension and contraction of the fibres. To Saint-Venant belongs the credit of bringing the problems of the torsion and flexure of beams under the general theory. Seeing the difficulty of obtaining general solutions, the pressing need for practical purposes of some theory that could be applied to the strength of structures, and the improbability of the precise mode of application of the load to the parts of any apparatus being known, he was led to reflect on the theories used for the solution of special problems before the discovery of the general equations. These reflexions led him to the discovery of the *semi-inverse* method of solution, which bears his name. Some part of the theory in vogue, and resting on special assumptions, may be true, at least in a large majority of cases. It may be possible, by

[1] *Exercices de Mathématiques*, 1828.

retaining some part of the data or conclusions of such a theory, to restrict the generality of the equations, and so obtain solutions—not indeed such as satisfy surface-conditions arbitrarily given, but such as satisfy sets of surface-conditions practically important.

The first problem to which Saint-Venant applied his method was that of the torsion of prisms, towards the theory of which he struggled from 1839 to 1855 when he gave it in his most famous memoir[1]. For this application he assumed the general character of the strain, viz.: that it consists of a distortion of the cross-sections combined with a simple twist about the axis; from this he deduced the differential equation and the boundary-condition that must be satisfied by the displacement parallel to the axis, and he shewed that the twisting couple may be of any given amount that produces no set, but the tractions, of which this couple is the resultant, must be applied to the end of the prism in a particular manner. In cases of symmetry the differential equation is Laplace's equation, and Saint-Venant made use of certain known solutions to discuss a large number of cases. The most important results are (i) that the sections do not remain plane, (ii) that Coulomb's torsion-formula is inexact, and requires for its correction a numerical factor depending on the shape of the cross-section. In the same memoir, and in a subsequent one[2], the same author applied his new method to the problem of flexure. He assumed that in a bent beam the axis, (or line of centroids of normal sections,) becomes a plane curve, and the extensions or contractions of longitudinal fibres vary as their distance from a certain plane through this axis, also that these fibres exert no mutual traction upon each other. The most important results are (i) that the stress across any section reduces to a transverse force and a bending couple, and the latter is proportional to the curvature of the axis, as given by the Bernoulli-Eulerian theory; (ii) that the normal sections do not remain plane, but the displacement in the direction of the axis contains a term, which satisfies an equation similar to that in the case of torsion with a different boundary-condition. The forces applied at the end may be any transverse force and bending couple, but these must be the resultants of tractions distributed over the end in a particular manner.

[1] *Mém. des savants étrangers*, XIV. 1855.

[2] 'Mémoire sur la flexion des prismes...' Liouville's *Journal*, I. 1856.

Both in the memoir on torsion and in that on flexure Saint-Venant enunciates the principle called by Prof. Pearson that of the "elastic equivalence of statically equipollent loads", according to which the strain at any point of a beam, whose length is several times its diameter, can be calculated without sensible error from the *resultant* force applied at its end, provided the point be not very near the end. We have already seen how the later researches of M. Boussinesq throw light on this principle.

In 1864 appeared Clebsch's *Theorie der Elasticität fester Körper*, a work which, in its present form, as edited by Saint-Venant, is the standard treatise on our subject. In this the problem of the equilibrium of beams is styled "*das de-Saint-Venantsche Problem*", and is treated in a more general manner. It appeared from Saint-Venant's researches that, alike in the cases of torsion and flexure, there is no stress in the normal section between fibres of the beam parallel to its length. Clebsch proposed to discover the general conditions under which this state of things will hold. He introduced this single condition into the equations of equilibrium, and proved that all the solutions that could thus be obtained fell into three classes characterised respectively by extension, torsion, and flexure. The equations to be satisfied are Saint-Venant's equations for the distortion of the sections.

The theory of torsion has received development at the hands of several writers, and we must mention especially the treatment of the subject in Thomson and Tait's *Natural Philosophy*. Here, for the first time, it was pointed out that the problem of the torsion of an elastic prism is mathematically identical with that of the motion of incompressible fluid in the same prism, rotating with angular velocity equal and opposite to the amount of the twist. This Hydrodynamical analogy, and the known method of solving problems in Hydrodynamics by means of conjugate functions, led to the discovery of a remarkable series of solutions of the torsion problem. The most important general results that can be gathered from this theory are (i) that the resistance of beams to torsion is seriously diminished by the existence of any concavity, or dent, or anything approaching to a reentrant angle in the surface, and (ii) that the correct formula[1] for the resistance of a beam to torsion,

[1] Saint-Venant, 'Sur une formule donnant approximativement le moment de torsion'. *Comptes Rendus*, LXXXVIII. 1879.

when this source of weakness is not present, makes this resistance very approximately equal to the product of an elastic constant, the fourth power of the area, the reciprocal of the moment of inertia about the axis, and the amount of the twist.

It is apparent that in the case of flexure the departure of the new from the old theory is not so glaring as in the case of torsion, the character of the resultant stress is given nearly enough by the old theory, it is however entirely at fault in describing the character of the strain, and consequently could not arrive at a correct estimate of the strength of a beam subject to flexure. This Saint-Venant's theory enables us to do more satisfactorily. An account of the theory, and its practical applications, is given in Saint-Venant's edition of the *Leçons de Navier* (1863). Most of these applications rest on an extension of the results for a beam supporting an isolated load, to the case of a continuously loaded beam. So far as I am aware, the only exact solution of the latter problem is that which has been recently given by Prof. Pearson[1], for a particular distribution of load. The extension to be made rests, in general, on the supposition that the linear dimensions of the cross-section of the beam are very small in comparison with its length, and they thus belong essentially to the theory of thin rods and wires. We shall therefore properly postpone our consideration of these extensions of Saint-Venant's theory, until we come to treat of that part of the subject.

[1] *Quarterly Journal*, 1889.

CHAPTER I.

ANALYSIS OF STRAIN[1].

1. WHENEVER, owing to any cause, changes take place in the relative positions of the parts of a body, the body is said to be strained—thus a stretched string, a compressed spring, a twisted wire, a vibrating bell, are bodies in a state of strain.

The part of our subject which deals with the analysis of strains—including their composition and resolution—is a branch of kinematics, and can be investigated from a purely geometrical point of view. For this purpose, we shall consider homogeneous strain as a method of transformation of geometrical figures, and shall then explain the connexion of this branch of geometry with our subject.

2. Homogeneous Strain.

Suppose we are given any figure (collection of points) in space, the points may be distributed either discretely or continuously, and points distributed continuously may form an

[1] The following among other authorities may be consulted:

Cauchy, *Exercices de Mathématiques*, *Année* 1827, the article 'Sur la condensation et la dilatation des corps solides'.

Saint-Venant, *Comptes Rendus* XXIV. 1847. 'Mémoire sur l'équilibre des corps solides, dans les limites de leur élasticité, et sur les conditions de leur résistance, quand les déplacements éprouvés par leurs points ne sont pas très-petits'.

Thomson and Tait, *Natural Philosophy*, vol. I. part I.

Sir W. Thomson, article 'Elasticity', *Encyclopædia Britannica*, reprinted in his *Mathematical and Physical Papers*, vol. III.

Todhunter and Pearson, *History of the Elasticity and Strength of Materials*, vol. I. especially arts. 1619 sq.

Weyrauch, *Theorie elastischer Körper*.

aggregate of one, two, or three dimensions, according as they lie upon a line, or upon a surface, or within a certain region of space. We shall suppose, in general, that the figure considered is a triply infinite series of points, filling a certain surface given in space. The position of any point in such a series is determined by means of its rectangular coordinates (x, y, z), referred to a Cartesian system of axes, and we shall in general suppose that the origin is one of the points of the series considered. Then the figure is said to be homogeneously strained when we make the new position of a point correspond to its old position in such a way that the coordinates of its new position are linear functions of the coordinates of its old position.

Let the equations of transformation be

$$\left.\begin{aligned} x_1 &= (1 + a_{11})\, x + a_{12} y + a_{13} z \\ y_1 &= a_{21} x + (1 + a_{22})\, y + a_{23} z \\ z_1 &= a_{31}\, x + a_{32}\, y + (1 + a_{33})\, z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Then x_1, y_1, z_1 are the coordinates, after strain, of the point which before strain was at (x, y, z).

The equations corresponding to (1), in two dimensions, represent a transformation such that the figure corresponding to a given one is similar to one of the orthographic projections of the original, and the characteristic property of such transformations is that parallel straight lines are transformed into parallel straight lines, and all the parts of any one straight line are equally extended. It is clear now that by such a transformation as (1) parallel planes are transformed into parallel planes, and thence it follows that this characteristic property of the transformation in two dimensions holds also for homogeneous strain in three dimensions. Again in orthographic projection any circle becomes an ellipse, and diameters of the circle at right angles to each other become conjugate diameters of the ellipse. In like manner it is at once seen from equations (1) that any sphere is changed by homogeneous strain into an ellipsoid, and three co-orthogonal diameters of the sphere are changed into three conjugate diameters of the ellipsoid. This ellipsoid is called the *strain-ellipsoid*. It follows that there is one set of co-orthogonal lines which remain such after strain, viz. these are the lines that become the principal axes of the ellipsoid. These lines are called the *principal axes* of the strain.

3. Extension and Shear.

Among homogeneous strains we shall note two in particular. In the first of these, which is called *simple extension*, lines parallel to a given direction are extended and all perpendicular lines are unaltered in length. It is clear that the equations of simple extension parallel to the x axis, when lines in this direction remain in it, are

$$x_1 = (1 + a_{11})\, x,\ y_1 = y,\ z_1 = z,$$

and in these a_{11} is the extension[1] parallel to x.

To see what is meant by shear, suppose all points in one plane to remain in that plane after strain, and in their primitive positions, and all points in any parallel plane to remain in their plane, but to be displaced in it in directions parallel to a given line in the first plane, and through distances proportional to their distances from that plane: *e.g.* suppose the planes $y = \text{const.}$ to move parallel to x, through distances proportional to y. This kind of strain is called *simple shear* of the planes y parallel to the axis x. The amount of sliding, per unit distance from the plane $y = 0$, is called the amount of the shear. It is clear that, if s be the amount of the shear, the equations of such a strain are

$$x_1 = x + sy,\ y_1 = y,\ z_1 = z.$$

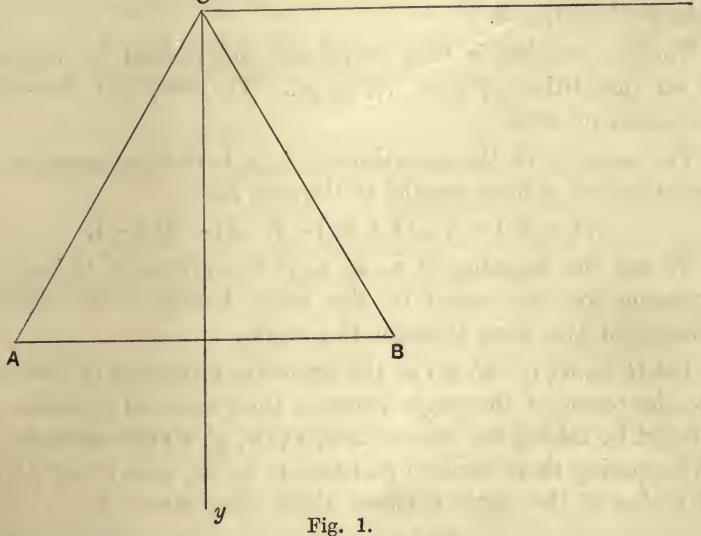

Fig. 1.

[1] By "extension" of a line we shall always mean the ratio of the increment of length to the original length. Contraction will be treated as negative extension.

In the figure let B be the new position of A, and suppose AB so chosen that the middle point of AB is on the axis of y, then the amount of the shear is 2 tan $\frac{1}{2}AOB$. The angle $\frac{1}{2}AOB$ is called the *angle of the shear*, and its tangent is half the amount of the shear.

4. Components of Strain.

The deformation of the figure will be completely known when we know the new length of every line in it. Since parallel lines are equally extended, we only need to know the new lengths of lines drawn through the origin.

Let l, m, n be the direction-cosines of a line drawn through the origin, and r its length, x, y, z the coordinates of its other extremity, so that

$$x = lr,\ y = mr,\ z = nr.$$

After strain let the point (x, y, z) come to (x_1, y_1, z_1). Then by writing lr, mr, nr for x, y, z in (1), squaring and adding we find the new length r_1 of the line given by the equation

$$r_1^2 = r^2 [1 + 2 (l^2\epsilon_1 + m^2\epsilon_2 + n^2\epsilon_3 + mns_1 + nls_2 + lms_3)] \ \ldots\ (2),$$

where

$$\left.\begin{aligned} \epsilon_1 &= a_{11} + \tfrac{1}{2}(a_{11}^2 + a_{21}^2 + a_{31}^2) \\ s_1 &= a_{23} + a_{32} + (a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33}) \end{aligned}\right\} \ldots\ldots\ldots\ldots (3),$$

and $\epsilon_2, \epsilon_3, s_2, s_3$ are to be found from these by cyclical interchanges of the suffixes 1, 2, 3.

The deformation is thus completely determined by means of the six quantities $\epsilon_1, \epsilon_2, \epsilon_3, s_1, s_2, s_3$. We shall call these the *components of strain.*

The meaning of the quantities $\epsilon_1, \epsilon_2, \epsilon_3$ is at once apparent, for the extensions of lines parallel to the axes are

$$\sqrt{(1 + 2\epsilon_1)} - 1,\ \sqrt{(1 + 2\epsilon_2)} - 1,\ \sqrt{(1 + 2\epsilon_3)} - 1.$$

To see the meaning of s_1, s_2, s_3, it is convenient to form an expression for the cosine of the angle between the strained positions of two lines through the origin.

Let (l, m, n), (l', m', n') be the primitive directions of the lines, then the cosine of the angle between their strained positions will be found by taking two points (x, y, z), (x', y', z') one on each line, and supposing their strained positions to be (x_1, y_1, z_1), (x_1', y_1', z_1'), the cosine of the angle between them after strain is

$$\frac{x_1x_1' + y_1y_1' + z_1z_1'}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}\ \sqrt{(x_1'^2 + y_1'^2 + z_1'^2)}}.$$

Now x_1, y_1, z_1 are given by (1) in terms of x, y, z, and x_1', y_1', z_1' are the same functions of x', y', z'; also $x : y : z = l : m : n$, and $x' : y' : z' = l' : m' : n'$. Hence we find, for the cosine in question, the value

$$\frac{ll'(1+2\epsilon_1)+mm'(1+2\epsilon_2)+nn'(1+2\epsilon_3)+(mn'+m'n)s_1+(nl'+n'l)s_2+(lm'+l'm)s_3}{\sqrt{[1+2(\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \tfrac{1}{2}s_1,\ \tfrac{1}{2}s_2,\ \tfrac{1}{2}s_3\}(lmn)^2]}\sqrt{[1+2(\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \tfrac{1}{2}s_1,\ \tfrac{1}{2}s_2,\ \tfrac{1}{2}s_3\}(l'm'n')^2]} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

In particular if the lines be the axes of x and y this reduces to

$$\frac{s_3}{\sqrt{(1+2\epsilon_1)}\sqrt{(1+2\epsilon_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Thus s_1, s_2, s_3 depend upon the angles between the strained positions of the lines initially coinciding with the axes.

Another way of looking at this matter is to suppose that the strain consists of a simple shear, say of the planes y parallel to the axis x.

Let the equations of the shear be

$$x_1 = x + sy,\ y_1 = y,\ z_1 = z.$$

Then the six components of strain are

$$\left.\begin{aligned} &\epsilon_1 = 0,\ \epsilon_2 = \tfrac{1}{2}s^2,\ \epsilon_3 = 0 \\ &s_1 = 0,\ s_2 = 0,\ \ s_3 = s \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (6),$$

so that s_3 is the amount of the shear. In the case of infinitesimal strain the shear of two lines initially at right angles is the cosine of the angle between them after strain, viz. this is the shear parallel to either line of planes perpendicular to the other.

5. The Strain-Quadric.

We shall call the quadric whose equation is

$$(\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \tfrac{1}{2}s_1,\ \tfrac{1}{2}s_2,\ \tfrac{1}{2}s_3\}(xyz)^2 = k \quad \ldots\ldots\ldots\ldots\ldots (7)$$

the *Strain-Quadric*. If r be the length of any line before strain, and r_1 the corresponding length after strain, then

$$r_1^2 = r^2 + 2k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

so that the square of every radius vector of the quadric is increased by the same amount.

Let the equation of the strain-quadric referred to its principal axes be

$$E_1x^2 + E_2y^2 + E_3z^2 = k \quad \ldots\ldots\ldots\ldots\ldots (9);$$

then, since the s components of strain are zero, it follows from (5) that the angles between lines, initially coinciding with the axes,

remain right angles after strain—so that the principal axes of the strain-quadric are the lines which remain co-orthogonal after strain, i.e. they are the same as the initial positions of the lines which become the principal axes of the strain-ellipsoid, or they are the principal axes of the strain defined in article 2.

The extensions of lines initially parallel to the axes of the strain-quadric are $\sqrt{(1+2E_1)}-1$, $\sqrt{(1+2E_2)}-1$, $\sqrt{(1+2E_3)}-1$, these are called the *principal extensions*, and we shall denote them by η_1, η_2, η_3. The equation of the strain-ellipsoid, referred to its principal axes in their strained position, is

$$x^2/(1+\eta_1)^2 + y^2/(1+\eta_2)^2 + z^2/(1+\eta_3)^2 = r^2,$$

where r is the radius of the sphere which is strained into the ellipsoid.

We now see that to specify a homogeneous strain we require to know the principal extensions, and the principal axes of the strain. In fact there are three lines of the figure initially at right angles, which are strained into lines at right angles, but in altered directions, and lengths initially parallel to these lines are extended in the ratios $1+\eta_1 : 1,\ 1+\eta_2 : 1,\ 1+\eta_3 : 1$.

6. Transformation of Strain-Components.

Suppose that a strain specified by $\epsilon_1, \epsilon_2, \epsilon_3, s_1, s_2, s_3$ is known, and that we wish to find the strain-components referred to a new system of co-orthogonal axes.

Let x', y', z' be the coordinates, referred to the new system, of a point whose coordinates, referred to the old system, are x, y, z, and let the scheme of transformation be

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

..... (10),

and let the components of the strain referred to the new axes be $\epsilon_1', \epsilon_2', \epsilon_3', s_1', s_2', s_3'$. Then, since the new and old lengths of a given line are independent of the system of axes, it follows that the strain-quadric (7) will be transformed to

$$(\epsilon_1', \epsilon_2', \epsilon_3', \tfrac{1}{2}s_1', \tfrac{1}{2}s_2', \tfrac{1}{2}s_3'\,\Delta xyz)^2 = k,$$

and thus $\epsilon_1'\ldots$ will be the coefficients in the transformed equation of the quadric (7). We thus obtain the equations

$$\left.\begin{aligned}\epsilon_1' &= \epsilon_1 l_1^2 + \epsilon_2 m_1^2 + \epsilon_3 n_1^2 + s_1 m_1 n_1 + s_2 n_1 l_1 + s_3 l_1 m_1,\\ s_1' &= 2\epsilon_1 l_2 l_3 + 2\epsilon_2 m_2 m_3 + 2\epsilon_3 m_2 n_3 + s_1(m_2 n_3 + m_3 n_2) + s_2(n_2 l_3 + n_3 l_2) + s_3(l_2 m_3 + l_3 m_2)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(11),$$

and the other components can be written down by cyclical interchanges of the suffixes of the l's, m's, and n's.

We remark that by a well-known theorem of Solid Geometry the quantities

$$\epsilon_1 + \epsilon_2 + \epsilon_3, \quad s_1^2 + s_2^2 + s_3^2 - 4(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2),$$
$$4\epsilon_1\epsilon_2\epsilon_3 + s_1 s_2 s_3 - \epsilon_1 s_1^2 - \epsilon_2 s_2^2 - \epsilon_3 s_3^2$$

are unaltered by the transformation of coordinates. These are called *invariants of the strain*, they are the coefficients in the discriminating cubic of the strain-quadric (7).

7. Examples. Extension and Shear.

We may utilise the properties of the strain-quadric to discuss the components of strain in particular cases. Of this we shall give two examples.

(1) Suppose the strain $(\epsilon_1, \epsilon_2, \epsilon_3, s_1, s_2, s_3)$ equivalent to a simple extension. Let e be the amount of the extension, and $\sqrt{(1+2E_1)} - 1 = e$, then referred to its principal axes the strain-quadric is

$$E_1 x^2 = k.$$

We thus find $$E_1 = \epsilon_1 + \epsilon_2 + \epsilon_3,$$

and the extension is therefore given by

$$e = \sqrt{\{1 + 2(\epsilon_1 + \epsilon_2 + \epsilon_3)\}} - 1 \quad \ldots\ldots\ldots\ldots\ldots(12),$$

where the positive square root is to be taken.

The conditions that the strain may be a simple extension are

$$\left.\begin{aligned}s_1^2 + s_2^2 + s_3^2 - 4(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2) &= 0\\ 4\epsilon_1\epsilon_2\epsilon_3 + s_1 s_2 s_3 - \epsilon_1 s_1^2 - \epsilon_2 s_2^2 - \epsilon_3 s_3^2 &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots(13).$$

Let l, m, n be the direction-cosines of the extended line, then we have

$$\frac{2\epsilon_1 l + s_3 m + s_2 n}{2l} = \frac{s_3 l + 2\epsilon_2 m + s_1 n}{2m} = \frac{s_2 l + s_1 m + 2\epsilon_3 n}{2n} = \kappa \text{ say},$$

where $\kappa = \epsilon_1 + \epsilon_2 + \epsilon_3$ is the root of the discriminating cubic of the strain-quadric that does not vanish. These equations determine the direction (l, m, n).

(2) Suppose the strain $(\epsilon_1, \epsilon_2, \epsilon_3, s_1, s_2, s_3)$ a simple shear of amount s. Then, if this be a shear of the planes y' parallel to the axis x', we shall have by (6) for the strain-quadric referred to (x', y', z')

$$\tfrac{1}{2}s^2y'^2 + sx'y' = k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

We therefore have

$$\left.\begin{aligned} \epsilon_1 + \epsilon_2 + \epsilon_3 &= \tfrac{1}{2}s^2 \\ s_1^2 + s_2^2 + s_3^2 - 4(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2) &= s^2 \\ 4\epsilon_1\epsilon_2\epsilon_3 + s_1s_2s_3 - \epsilon_1 s_1^2 - \epsilon_2 s_2^2 - \epsilon_3 s_3^2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots (15),$$

and the conditions that the strain may be a simple shear are the third of (15) and

$$s_1^2 + s_2^2 + s_3^2 - 4(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2) = 2(\epsilon_1 + \epsilon_2 + \epsilon_3) \ldots\ldots (16).$$

The amount of the shear s is

$$s = \sqrt{\{s_1^2 + s_2^2 + s_3^2 - 4(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2)\}} \ldots\ldots\ldots\ldots (17).$$

The equation of the strain-quadric, referred to its principal axes, is

$$\frac{s^2}{4}(x^2 + y^2) + \frac{s\sqrt{(s^2+4)}}{4}(x^2 - y^2) = k \ldots\ldots\ldots\ldots\ldots (18).$$

If this be written $E_1x^2 + E_2y^2 = k$,

then it is easily verified that

$$\sqrt{(1 + 2E_1)}\sqrt{(1 + 2E_2)} = 1,$$

and

$$\sqrt{(1 + 2E_1)} - \sqrt{(1 + 2E_2)} = s,$$

or if η_1, η_2 be the two principal extensions that do not vanish,

$$(1 + \eta_1)(1 + \eta_2) = 1, \quad \eta_1 - \eta_2 = s \ldots\ldots\ldots\ldots\ldots (19).$$

This shews that shear is a state of plane strain which involves no change in the volume of any part of the figure, and that its amount is equal to the difference of the two principal extensions.

To find the principal axes of the shear, referred to the axes of (x', y'), we suppose one of these axes to make an angle θ with the axis x', then

$$\frac{\frac{1}{2}s\sin\theta}{\cos\theta} = \frac{\frac{1}{2}s\cos\theta + \frac{1}{2}s^2\sin\theta}{\sin\theta} = \frac{s^2 \pm s\sqrt{(s^2+4)}}{4},$$

whence

$$\tan\theta = \frac{s}{2} \pm \sqrt{\left(\frac{s^2}{4} + 1\right)}.$$

Let α be the angle of the shear, then $\frac{1}{2}s = \tan\alpha$, so that

$$\theta = \tfrac{1}{4}\pi + \tfrac{1}{2}\alpha, \text{ or } \tfrac{3}{4}\pi + \tfrac{1}{2}\alpha.$$

Thus, in figure 2, the principal axes of the shear are the internal and external bisectors of the angle AOx, these are the lines $O1$, $O2$, and there is extension of lines parallel to $O1$ and contraction of lines parallel to $O2$.

After the strain these lines will not retain their primitive directions, but we may find the angle through which they are rotated.

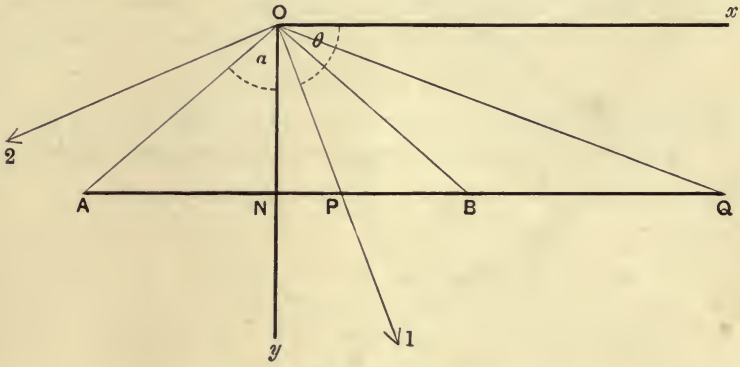

Fig. 2.

In the figure let P be a point on one principal axis before strain, and Q its strained position, ON the perpendicular on PQ. Suppose $ON = 1$, then $PQ = s = 2 \tan \alpha$, and $\tan \theta = (1 + \sin \alpha) \sec \alpha$.

Hence

$$NQ = \cot \theta + s = \cos \alpha / (1 + \sin \alpha) + 2 \tan \alpha = \tan \alpha + \sec \alpha,$$

and $$\text{angle } POQ = \tan^{-1} \frac{1 + \sin \alpha}{\cos \alpha} - \tan^{-1} \frac{\cos \alpha}{1 + \sin \alpha} = \alpha.$$

Thus the principal axes of the shear are rotated through an angle equal to the angle of the shear. It is clear that after the strain the figure can be turned back through this angle without any alteration of the length of any line in it, and the simple shear combined with this rotation is called a *pure shear*.

In pure shear lines parallel to one of the principal axes of the shear are extended, and lines parallel to the other principal axis are contracted, and since the principal extensions are connected by the relation

$$(1 + \eta_1)(1 + \eta_2) = 1$$

given in equation (19), we get the following representation of pure shear:

Let $ABCD$ be a rhombus, whose diagonals are in the ratio $1+\eta_1 : 1+\eta_2$, and are in the direction of the principal axes of the

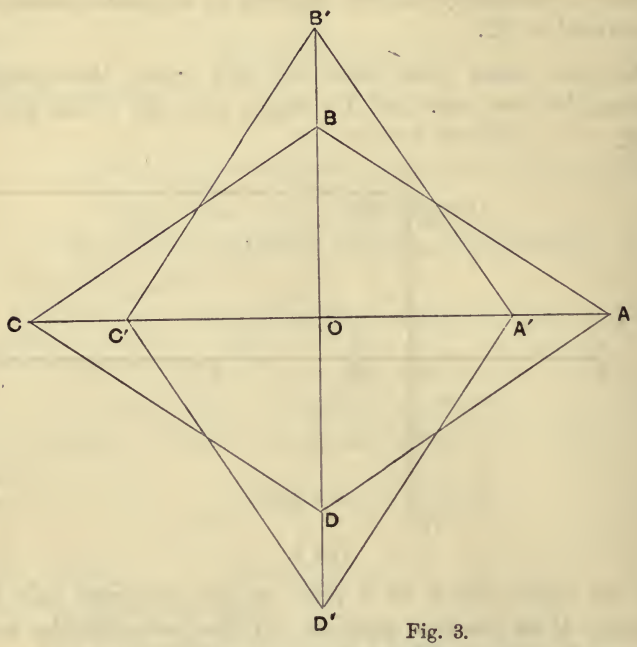

Fig. 3.

shear, and let $A'B'C'D'$ be an equal rhombus, with its corresponding diagonals at right angles to those of $ABCD$; then by the pure shear, consisting of contraction along AC and extension along BD, the first rhombus will be transformed into the second.

The reader should find no difficulty in verifying the following methods of producing any homogeneous strain :

(1) Any such strain can be produced in a figure by a shear parallel to one axis of planes perpendicular to another, a uniform extension perpendicular to the plane of the two axes, a uniform extension of all lines of the figure, and a rotation.

(2) Any such strain can be produced by three shears each of which is a shear parallel to one axis of planes perpendicular to another, a uniform extension of all lines of the figure, and a rotation.

8. Pure Strain.

In general a strain is said to be *pure* when the principal axes of the strain-ellipsoid are lines which retain their primitive

directions. In this case the principal axes of the strain-quadric are lines which retain their primitive directions, and lines of the figure parallel to these axes are simply extended in certain ratios.

Suppose the strain represented by equations (1) is a pure strain, and let $\alpha-1$, $\beta-1$, $\gamma-1$ be the principal extensions. Let ξ_1, η_1, ζ_1 be the coordinates after strain of a point whose coordinates before strain are ξ, η, ζ, the axes of (ξ, η, ζ) being the principal axes of the strain. Then, since all lines parallel to the axes are elongated in the ratios $\alpha:1$, $\beta:1$, $\gamma:1$, we have

$$\xi_1=\alpha\xi,\ \eta_1=\beta\eta,\ \zeta_1=\gamma\zeta.$$

Let the principal axes, viz. the axes of (ξ, η, ζ), be given with reference to the axes of (x, y, z) by the scheme

	x	y	z
ξ	l_1	m_1	n_1
η	l_2	m_2	n_2
ζ	l_3	m_3	n_3

and let x_1, y_1, z_1 be the coordinates, after strain, of the point, whose coordinates, before strain, are x, y, z; then

$$\begin{aligned}x_1&=l_1\xi_1+l_2\eta_1+l_3\zeta_1\\&=l_1\alpha\xi+l_2\beta\eta+l_3\gamma\zeta\\&=l_1\alpha(l_1x+m_1y+n_1z)+l_2\beta(l_2x+m_2y+n_2z)+l_3\gamma(l_3x+m_3y+n_3z).\end{aligned}$$

The coefficient of y in the expression for x_1, *i.e.* the coefficient a_{12}, is

$$\alpha l_1m_1+\beta l_2m_2+\gamma l_3m_3,$$

and we should find the same value for a_{21}. We should find in like manner $a_{23}=a_{32}$, $a_{31}=a_{13}$.

Thus, if the strain be pure, we have the relations

$$a_{23}=a_{32},\ a_{31}=a_{13},\ a_{12}=a_{21}\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Conversely we may shew that, if the equations (20) hold good, the strain is pure. Suppose the strain given by equations (1), and write

$$\left.\begin{array}{lll}a_{11}=1+e, & a_{22}=1+f, & a_{33}=1+g\\ a_{23}+a_{32}=a, & a_{31}+a_{13}=b, & a_{12}+a_{21}=c\\ a_{32}-a_{23}=2\varpi_1, & a_{13}-a_{31}=2\varpi_2, & a_{21}-a_{12}=2\varpi_3\end{array}\right\}\ldots\ldots(21).$$

The displacements of any point are[1]

$$\left.\begin{aligned} x_1 - x &= ex + \tfrac{1}{2}cy + \tfrac{1}{2}bz - y\varpi_3 + z\varpi_2 \\ y_1 - y &= \tfrac{1}{2}cx + fy + \tfrac{1}{2}az - z\varpi_1 + x\varpi_3 \\ z_1 - z &= \tfrac{1}{2}bx + \tfrac{1}{2}ay + gz - x\varpi_2 + y\varpi_1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (22).$$

If ϖ_1, ϖ_2, ϖ_3 be separately equal to zero, the resultant displacement of any point is along the normal to that quadric of the family

$$(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c\Sigma xyz)^2 = \text{const.} \ldots\ldots\ldots\ldots (23)$$

which passes through the point. Hence points on any principal axis of these quadrics remain in it, *i.e.* the three co-orthogonal lines that remain co-orthogonal after strain retain their primitive directions, and the strain is pure.

Thus the necessary and sufficient conditions that the strain may be pure are equations (20), or in the notation of (21)

$$\varpi_1 = 0, \quad \varpi_2 = 0, \quad \varpi_3 = 0.$$

It is shewn in Art. 10 that, when the displacements are small, these quantities ϖ_1, ϖ_2, ϖ_3 are the component rotations of any small part of the figure about axes parallel to the coordinate axes, and for this reason pure strain is often described as irrotational.

9. The Elongation-Quadric.

The quadric (23) is called the *Elongation-Quadric.* Let P be any point (x, y, z) of the figure, which is transformed to $P_1(x_1, y_1, z_1)$ by the strain, then, if we define the elongation of OP in direction OP to be the projection of P_1P on OP, this is

$$(x_1 - x)\frac{x}{OP} + (y_1 - y)\frac{y}{OP} + (z_1 - z)\frac{z}{OP},$$

and by (22), whether ϖ_1, ϖ_2, ϖ_3 vanish or not, this is

$$\frac{1}{OP}(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c\Sigma xyz)^2.$$

Thus the *rate* of elongation of OP in direction OP is found by dividing this by OP, or, if l, m, n be the direction-cosines of OP, this rate of elongation is

$$(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c\Sigma lmn)^2.$$

[1] For an account of the kind of symmetry possessed by these expressions see the Note on Double Suffix Notations at the end of chapter III.

Hence the rate of elongation of any radius r of the quadric (23), measured along r, is inversely as the square of this radius.

Let the equation of the elongation-quadric referred to its principal axes be

$$e_1x^2+f_1y^2+g_1z^2=k,$$

and suppose the strain pure, then the equations of strain referred to principal axes are by (22)

$$x_1=(1+e_1)x,\ y_1=(1+f_1)y,\ z_1=(1+g_1)z,$$

and we thus see that, when the strain is pure, the quantities e_1, f_1, g_1 are the principal extensions.

In general, by changing to new rectangular axes the following quantities are unaltered, viz :—

$$\left.\begin{array}{c} e+f+g \\ a^2+b^2+c^2-4(fg+ge+ef) \\ 4efg+abc-ea^2-fb^2-gc^2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (24).$$

From the two first of these we can deduce that

$$\tfrac{1}{2}(a^2+b^2+c^2)+(e^2+f^2+g^2)$$

is also an invariant for orthogonal transformations.

10. Composition of Strains.

Suppose a figure transformed by the homogeneous strain given by equations (1), and the new figure transformed again by homogeneous strain. Let the point (x, y, z) come to (x_1, y_1, z_1) after the first strain, and (x_1, y_1, z_1) come to (x_2, y_2, z_2) after the second strain, and let the equations of the second transformation be

$$\left.\begin{array}{l} x_2=(1+b_{11})x_1+b_{12}y_1+b_{13}z_1 \\ y_2=b_{21}x_1+(1+b_{22})y_1+b_{23}z_1 \\ z_2=b_{31}x_1+b_{32}y_1+(1+b_{33})z_1 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (25).$$

If we write the equations (1) and (25) in Prof. Cayley's matrix-notation

$$(x_1, y_1, z_1)=\begin{vmatrix} 1+a_{11}, & a_{12}, & a_{13} \\ a_{21}, & 1+a_{22}, & a_{23} \\ a_{31}, & a_{32}, & 1+a_{33} \end{vmatrix}(x, y, z)\ldots\ldots(1),$$

$$(x_2, y_2, z_2)=\begin{vmatrix} 1+b_{11}, & b_{12}, & b_{13} \\ b_{21}, & 1+b_{22}, & b_{23} \\ b_{31}, & b_{32}, & 1+b_{33} \end{vmatrix}(x_1, y_1, z_1)\ldots(25),$$

we can write the equations of transformation, that express x_2, y_2, z_2 in terms of x, y, z, in the form

$$(x_2, y_2, z_2) = \begin{pmatrix} 1+c_{11}, & c_{12}, & c_{13} \\ c_{21}, & 1+c_{22}, & c_{23} \\ c_{31}, & c_{32}, & 1+c_{33} \end{pmatrix} (x, y, z) \quad \ldots(26),$$

in which

$$\left.\begin{aligned} c_{11} &= (1+a_{11})(1+b_{11}) + a_{21}b_{12} + a_{31}b_{13} \\ c_{12} &= a_{12}(1+b_{11}) + (1+a_{22})\,b_{12} + a_{32}b_{13} \\ c_{13} &= a_{13}(1+b_{11}) + a_{23}b_{12} + (1+a_{33})\,b_{13} \end{aligned}\right\} \quad \ldots\ldots\ldots(27),$$

and the other coefficients c can be written down by symmetry. Of this there are several interesting particular cases—

(i) Suppose the component strains (1) and (25) are pure, the resultant strain is not in general a pure strain. We have for example

$$c_{12} = a_{12} + b_{12} + a_{12}b_{11} + a_{22}b_{12} + a_{32}b_{13},$$
$$c_{21} = a_{21} + b_{21} + a_{11}b_{21} + a_{21}b_{22} + a_{31}b_{23},$$

and these will not become equal on putting $a_{12} = a_{21}, \ldots$.

(ii) Suppose the strain (1) a pure strain, and the axes of (x, y, z) the principal axes, so that the elongation-quadric is $e_1x^2 + f_1y^2 + g_1z^2 = k$, and suppose the substitution (25) equivalent to a simple rotation θ, about an axis whose direction-cosines are l, m, n. Then we know that the equations corresponding to (25) are three of the form

$$x_2 - x_1 = \sin\theta\,(z_1m - y_1n) + 2\sin^2\frac{\theta}{2}\{(x_1l + y_1m + z_1n)\,l - x_1\},$$

(see Minchin's *Statics*, 3rd edition, vol. II. p. 104).

Thus the coefficients in the substitution (25) are given by equations of the form

$$\left.\begin{aligned} 1 + b_{11} &= 1 + 2(l^2 - 1)\sin^2\frac{\theta}{2} \\ b_{12} &= -n\sin\theta + 2lm\sin^2\frac{\theta}{2} \\ b_{13} &= m\sin\theta + 2nl\sin^2\frac{\theta}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(28),$$

where the other b's can be put down by symmetry.

Now writing down the coefficients of (1) in the forms

$$\left.\begin{aligned} a_{11} = e_1, \quad a_{22} = f_1, \quad a_{33} = g_1 \\ a_{23} = a_{32} = 0, \quad a_{31} = a_{13} = 0, \quad a_{12} = a_{21} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(29),$$

we can write the coefficients of (26) in such forms as

$$\left.\begin{aligned} 1 + c_{11} &= (1 + e_1)(1 + 2\overline{l^2 - 1}\sin^2\tfrac{1}{2}\theta), \\ c_{32} &= (1 + f_1)(l\sin\theta + 2mn\sin^2\tfrac{1}{2}\theta), \\ c_{23} &= (1 + g_1)(-l\sin\theta + 2mn\sin^2\tfrac{1}{2}\theta \end{aligned}\right\} \ldots\ldots (30).$$

It may be verified analytically, and is geometrically obvious, that the six components of strain, corresponding to the substitution (28), vanish identically, and that the six components of strain, corresponding to equations such as (30), are respectively

$$\epsilon_1 = \tfrac{1}{2}[(1 + e_1)^2 - 1], \quad \epsilon_2 = \tfrac{1}{2}[(1 + f_1)^2 - 1], \quad \epsilon_3 = \tfrac{1}{2}[(1 + g_1)^2 - 1],$$
$$s_1 = 0, \quad s_2 = 0, \quad s_3 = 0.$$

In general we note that

$$c_{32} - c_{23} = 2l\sin\theta + (f_1 + g_1)\, l\sin\theta + (f_1 - g_1)\, 2mn\sin^2\tfrac{1}{2}\theta \ldots (31).$$

Now it is geometrically obvious that any homogeneous strain is a pure strain combined with a certain rotation. Also by comparing (28) with (22) we see that, when the equations of transformation, such as (22), correspond to a simple rotation θ, the quantities $\varpi_1, \varpi_2, \varpi_3$ are the products of $\sin\theta$ and the cosines of the angles which the axis of rotation makes with the axes of coordinates. It appears however from (31) that, when the equations correspond to a pure strain combined with a rotation, the quantities $\varpi_1, \varpi_2, \varpi_3$ no longer have this meaning, unless the pure strain be indefinitely small.

It may be shewn that, if P_1, P_2, P_3 be the areas of the projections of any closed curve on the coordinate planes, then the line-integral

$$\int (x_1 - x)\, dx + (y_1 - y)\, dy + (z_1 - z)\, dz$$

taken round the curve is $2P_1\varpi_1 + 2P_2\varpi_2 + 2P_3\varpi_3$. From which it follows that we may interpret ϖ_1 as half the line-integral of the tangential displacement round a closed curve of unit area in the plane yz, with similar interpretations for ϖ_2 and ϖ_3. The proof is left to the reader.

(iii) When, in equations (1) and (25), all the coefficients $a_{11}, \ldots, b_{11}, \ldots$ are infinitesimal, the displacement of every point of the figure will be infinitesimal and the equations giving the resultant displacements reduce to

$$(x_2 - x,\ y_2 - y,\ z_2 - z) = \begin{vmatrix} a_{11} + b_{11}, & a_{12} + b_{12}, & a_{13} + b_{13} \\ a_{21} + b_{21}, & a_{22} + b_{22}, & a_{23} + b_{23} \\ a_{31} + b_{31}, & a_{32} + b_{32}, & a_{33} + b_{33} \end{vmatrix} (x, y, z) \quad \ldots (32);$$

so that the resultant strains will be found from the component strains by simple addition. In particular we notice that if the component strains be pure the resultant strain is also pure.

(iv) A case of great importance is that of the composition of two shears, especially of two infinitesimal shears of perpendicular planes.

With the notation of art. 8 the equations of displacement may be written

$$x_1 - x = \tfrac{1}{2}bz, \quad y_1 - y = \tfrac{1}{2}az, \quad z_1 - z = \tfrac{1}{2}bx + \tfrac{1}{2}ay.$$

The elongation quadric is

$$ayz + bzx = \text{const.}$$

and its discriminating cubic takes the form

$$\kappa^3 - \kappa \tfrac{1}{4}(a^2 + b^2) = 0,$$

so that the strain is equivalent to equal extension and contraction, each $\frac{1}{2}\sqrt{(a^2+b^2)}$, along two lines at right angles, *i.e.* to a shear of amount $\sqrt{(a^2+b^2)}$.

11. Infinitesimal Strains.

The case where the displacements are infinitesimal is the most important for the mathematical theory of Elasticity. In this case the six quantities e, f, g, a, b, c of equations (21) are all very small, and ultimately identical with the six quantities $\epsilon_1, \epsilon_2, \epsilon_3, s_1, s_2, s_3$ of equations (3), so that the coefficients of $x^2, y^2, z^2, yz, zx, xy$ in the elongation-quadric are the six components of strain. The strain-quadric and the elongation-quadric in this case coincide. In the same case, the quantities $\varpi_1, \varpi_2, \varpi_3$ are the components of the infinitesimal rotation of the principal axes of the strain-quadric about the coordinate axes, as they pass from their positions before, to their positions after, strain.

The strains may be small, but the displacements finite. In this case all the quantities $\epsilon_1, \epsilon_2, \epsilon_3, s_1, s_2, s_3$ must be small, but the coefficients $a_{11}, \ldots$ of equations (1) need not be small. Thus for small strain it is not necessary that $e, f, g, a, b, c, \varpi_1, \varpi_2, \varpi_3$ be small. If however e, f, g, a, b, c be very small, then the strain will not be infinitesimal unless $\varpi_1, \varpi_2, \varpi_3$ are small also.

In the case of infinitesimal displacements, we may analyse the strain represented by the six components e, f, g, a, b, c. The quantities e, f, g are, as in art. 4, extensions of lines initially parallel to the coordinate axes, and the quantity a is a shear of

the planes $y = \text{const.}$ parallel to the axis z, or of the planes $z = \text{const.}$ parallel to the axis y; in like manner b and c can be interpreted as shears.

The elongation-quadric or strain-quadric for the same case is

$$(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c\,\Big)(xyz)^2 = k,$$

and the extension of a radius r in any direction is k/r^2.

For the transformation of strain-components to new rectangular axes, we have, with a notation similar to that of art. 10, the quadric

$$(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c\,\Big)(xyz)^2 = k$$

transformed into $(e', f', g', \tfrac{1}{2}a', \tfrac{1}{2}b', \tfrac{1}{2}c'\,\Big)(xyz)^2 = k,$

and thus we have 6 equations of the forms

$$\left.\begin{aligned} e' &= el_1^2 + fm_1^2 + gn_1^2 + am_1n_1 + bn_1l_1 + cl_1m_1, \\ a' &= 2el_2l_3 + 2fm_2m_3 + 2gn_2n_3 + a(m_2n_3 + m_3n_2) + b(n_2l_3 + n_3l_2) + c(l_2m_3 + l_3m_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(33).$$

From these we might deduce the invariants (24). Of these invariants the first, $e+f+g$, is the *cubical dilatation, i.e.* it is the ratio of the increment of volume of any part of the figure to the original volume, and the invariant

$$e^2 + f^2 + g^2 + \tfrac{1}{2}(a^2 + b^2 + c^2)$$

is the ratio of the integral

$$\iiint \{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2\}\, dx\,dy\,dz,$$

through any small volume possessing kinetic symmetry about the point (x, y, z), to the moment of inertia of the same volume with respect to any plane through (x, y, z)[1].

We can also put down two other invariants. These are

$$\left.\begin{aligned} &\varpi_1^2 + \varpi_2^2 + \varpi_3^2, \\ &(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c\,\Big)(\varpi_1\varpi_2\varpi_3)^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(34).$$

The first is geometrically obvious, since the resultant rotation is independent of the choice of axes, and the second is analytically obvious, since $(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c\,\Big)(xyz)^2$ transforms into

$$(e', f', g', \tfrac{1}{2}a', \tfrac{1}{2}b', \tfrac{1}{2}c'\,\Big)(x'y'z')^2,$$

whenever $x^2 + y^2 + z^2$ transforms into $x'^2 + y'^2 + z'^2$.

The results of art. 7 for very small shear are that the equation of the elongation-quadric for very small shear c of the planes $y = \text{const.}$ parallel to x, or of the planes $x = \text{const.}$ parallel to y is

[1] Betti, 'Teoria della Elasticita'. *Il Nuovo Cimento, Serie* 2, VII. 1872.

$cxy = k$, and referred to its principal axes this is $\frac{1}{2}c(x^2 - y^2) = k$. In other words equal extension and contraction e, along two lines at right angles, are equivalent to a shear of amount $2e$, of the planes parallel to the bisectors of the principal axes, and the angle of the shear is equal to half the amount of the shear. The axes of the shear become lines inclined to one another at an angle, whose complement is equal to the amount of the shear, so that, as remarked in art. 4, the shear of two rectangular lines, when very small, is the cosine of the angle between them after strain. The shear of two rectangular lines is often spoken of as a shear of their plane.

12. Strain in a body.

Now regarding a body as continuously filling a region of space, there will be a particle of the body at any point P, whose co-ordinates are x, y, z. Suppose $x+\xi, y+\eta, z+\zeta$ are the coordinates of a neighbouring point Q. If any system of forces be applied to the body, it will in general be deformed. In the deformation that takes place, let the particle, that was at (x, y, z) be displaced to $(x+u, y+v, z+w)$. The quantities u, v, w are the component displacements of this point of the body, and they must be, in general, continuous functions of the position of the point, as otherwise two points, originally very near together, would not remain near together and the body would be ruptured. Suppose u', v', w' are the component displacements of the point Q, then these are the same functions of $x+\xi, y+\eta, z+\zeta$ that u, v, w are of x, y, z, and we may expand u', v', w' in powers of ξ, η, ζ by Maclaurin's Theorem, and obtain for the coordinates of the new position of Q such quantities as

$$x + \xi + u + \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} + \text{terms of higher order in } \xi, \eta, \zeta,$$

so that the coordinates of Q relative to the new position of P are ultimately

$$\left.\begin{aligned} &\xi\left(1+\frac{\partial u}{\partial x}\right)+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z}, \\ &\xi\frac{\partial v}{\partial x}+\eta\left(1+\frac{\partial v}{\partial y}\right)+\zeta\frac{\partial v}{\partial z}, \\ &\xi\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+\zeta\left(1+\frac{\partial w}{\partial z}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (35),$$

where squares and higher powers of ξ, η, ζ are neglected.

These expressions are the coordinates of Q relative to P after the deformation, and they may be compared with the right-hand sides of equations (1). If then we take a notation similar to (21), viz.

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial x}, & f &= \frac{\partial v}{\partial y}, & g &= \frac{\partial w}{\partial z}, \\ a &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, & b &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, & c &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \\ 2\varpi_1 &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, & 2\varpi_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, & 2\varpi_3 &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{aligned}\right\} \quad (36),$$

we find that the component displacements of Q, when P is regarded as held fixed, are

$$\left.\begin{aligned} e\xi + \tfrac{1}{2}c\eta + \tfrac{1}{2}b\zeta - \eta\varpi_3 + \zeta\varpi_2 \\ \tfrac{1}{2}c\xi + f\eta + \tfrac{1}{2}a\zeta - \zeta\varpi_1 + \xi\varpi_3 \\ \tfrac{1}{2}b\xi + \tfrac{1}{2}a\eta + g\zeta - \xi\varpi_2 + \eta\varpi_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (37).$$

Thus the particles in the neighbourhood of P will come into new positions, which are derived from their original positions by a homogeneous strain.

A body deformed in any manner is said to be strained, and we see that, if the displacements be continuous functions of position, the strain about any point is sensibly homogeneous.

The relative displacements will be indefinitely small if all the first differential coefficients of u, v, w be indefinitely small. In this case the quantities $e, f, g, a, b, c, \varpi_1, \varpi_2, \varpi_3$ are all indefinitely small. We recognize that $\varpi_1, \varpi_2, \varpi_3$ are the component rotations of the matter about P, moving as if rigid, and thus that the most general system of small relative displacements of the matter about any point can be analysed into a small rotation, and a small pure strain. We also recognize that the quantities e, f, g are extensions of the matter lying originally in lines through P parallel to the axes, and that a, b, c are shears of planes through P parallel to the coordinate planes. The six quantities e, f, g, a, b, c are called the components of strain, and we know that they are equivalent to three simple extensions of all lines parallel to the principal axes of the elongation-quadric. All the results of art. 11 in regard to invariants, the cubical dilatation, the resultant

rotation &c. hold for the matter about any point. For convenience we state here that

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

is the *cubical dilatation* of the matter about the point (x, y, z).

If the strain be pure, $\varpi_1 = 0$, $\varpi_2 = 0$, $\varpi_3 = 0$, or we have

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y},$$

so that there exists a function ϕ, such that

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}.$$

The function ϕ is called the "displacement-potential." Its existence is confined to the case where the strain is pure.

If the displacements be finite, the deformation of the body in the neighbourhood of any point P can still be expressed by six components of strain. Let r be the unstrained length of any short line PQ of the body, r_1 its length after strain, and l, m, n the direction-cosines of PQ before the strain, then, as in art. 4,

$$(r_1^2 - r^2)/r^2 = 2(l^2\epsilon_1 + m^2\epsilon_2 + n^2\epsilon_3 + mns_1 + nls_2 + lms_3) \ldots (39),$$

where

$$\left.\begin{aligned} \epsilon_1 &= \frac{\partial u}{\partial x} + \tfrac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\ s_1 &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \end{aligned}\right\} \ldots\ldots (40),$$

and ϵ_2, ϵ_3, s_2, s_3 are to be found by cyclical interchanges of the letters (x, y, z) and (u, v, w). These equations are deduced from (35) in the same way as equations (3) from equations (1).

The necessary and sufficient conditions that the strain be everywhere small are that ϵ_1, ϵ_2, ϵ_3, s_1, s_2, s_3 be everywhere small.

All the conclusions of arts. 5 and 6, with regard to the transformation of strain-components, invariants, and the properties of the strain-quadric, hold for the strain of the matter about any point, and likewise the conclusions of art. 7 with regard to the analysis of particular strains.

It is easily verified that the cubical dilatation in the general case is

$$\sqrt{[1+2(\epsilon_1+\epsilon_2+\epsilon_3)+(4\epsilon_2\epsilon_3+4\epsilon_3\epsilon_1+4\epsilon_1\epsilon_2-s_1^2-s_2^2-s_3^2)} \\ +2(4\epsilon_1\epsilon_2\epsilon_3+s_1s_2s_3-\epsilon_1s_1^2-\epsilon_2s_2^2-\epsilon_3s_3^2)]-1,$$

and, in case the strain is infinitesimal, this is ultimately

$$\epsilon_1+\epsilon_2+\epsilon_3,$$

whether the displacements be small or not.

CHAPTER II.

ANALYSIS OF STRESS[1].

13. Stress at a point.

When a solid is strained forces will in general be called into play which resist the strain, we propose to investigate the character of the system of forces thus arising. Any molecule of the solid is regarded as exerting upon any other an action depending on the state and configuration of the system of molecules, and the second exerts an equal and opposite reaction upon the first. Consider any plane drawn in the solid, the molecules on the one side of the plane exert upon those on the other side forces in lines which cross the plane. Let us fix our attention on an element dS of the plane. The forces whose lines of action cross dS can be reduced to a resultant at the centroid of dS and a couple. The order of magnitude of the couple in the linear dimensions of dS is higher by unity than that of the force, and therefore, when the element dS is infinitesimal, the couple may be left out of account.

[1] The following among other authorities may be consulted:

Cauchy, *Exercices de Mathématiques*, *Année* 1827, the article 'De la pression ou tension dans un corps solide', and *Année* 1828, the article 'De la pression ou tension dans un système de points matériels'.

Lamé, *Leçons sur la théorie mathématique de l'élasticité des corps solides.*

Thomson and Tait, *Natural Philosophy*, vol. I. part II.

Sir W. Thomson, *Mathematical and Physical Papers*, vol. III.

Basset, *Hydrodynamics*, ch. II, XX.

Todhunter and Pearson, *History of the Elasticity and Strength of Materials*, vol. I, especially Appendix, Note B.

Maxwell, *Electricity and Magnetism.*

British Association Report, 1885. Sir W. Thomson's *Address to Section A.*

The forces have therefore a single resultant, and this resultant constitutes the traction across dS. Let this traction be resolved into three components in lines mutually at right angles, viz. NdS in the normal to the plane, and TdS and $T'dS$ parallel to two rectangular lines in the plane. Then N, T, T' are called the components of stress across dS.

Now let us take any point O of the solid and through it draw three planes at right angles to each other, and take these as coordinate planes in a system of rectangular coordinates (x, y, z). The stress across an element of the plane x at the point O will have components X_x parallel to x, Y_x parallel to y, and Z_x parallel to z. The first of these X_x is normal to the plane x, and the other two tangential to it, and these are the components of the traction exerted by the matter on the side x positive upon the matter on the side x negative. The normal stress is reckoned positive when it is a tension and negative when it is a pressure.

In like manner the stresses on the other two planes have components X_y, Y_y, Z_y, and X_z, Y_z, Z_z, the capitals indicating the direction of the stress-components, and the suffixes the planes across which they act.

We may shew that a knowledge of the stresses across these three planes at O is sufficient to enable us to determine the stress across any other plane through O.

Draw a plane very near to O in direction normal to a line whose direction-cosines are l, m, n, and let Δ be the area cut out on this plane by the three coordinate planes, and consider the equilibrium of the elementary tetrahedron of the solid whose faces are Δ and $l\Delta$, $m\Delta$, $n\Delta$. Let F, G, H be the components across Δ of the traction per unit area exerted by the matter on the side of the plane outside the tetrahedron upon that on the other side. The forces acting on the matter within the tetrahedron are the bodily forces, and the tractions across its four faces, of which the former are estimated per unit mass, and the latter per unit area. When the tetrahedron is indefinitely diminished, the bodily forces multiplied by the mass within the tetrahedron will give us terms in the equations of equilibrium or small motion of the order of the cube of the linear dimensions; the surface-tractions, multiplied each by the area of the face across which it acts, will give us terms of the order of the square of the linear dimensions, and the

former terms are in the limit negligible in comparison with the latter. Thus for the equilibrium of the elementary tetrahedron we have, by resolving parallel to x,

$$-X_x l\Delta - X_y m\Delta - X_z n\Delta + F\Delta = 0,$$

and two similar equations by resolving parallel to y and z. These are equivalent to

$$\left.\begin{aligned} F &= lX_x + mX_y + nX_z \\ G &= lY_x + mY_y + nY_z \\ H &= lZ_x + mZ_y + nZ_z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which determine F, G, H, the components of traction across any plane, in terms of the direction of the plane and the stresses across the three coordinate planes.

14. Equations of equilibrium and small motion.

From these expressions we can obtain the general equations of equilibrium of the solid. Let X, Y, Z be the bodily forces per unit mass acting at any point of the solid, and ρ its density, so that the components of the external force applied to any element of volume $dxdydz$ are $\rho X dxdydz$, $\rho Y dxdydz$, $\rho Z dxdydz$, and let dS be an element of an arbitrary closed surface S drawn in the solid, and l, m, n the direction-cosines of the normal to dS drawn outwards. Then the sum of the components in any direction of all the forces applied to the part of the solid within S must be equal to zero. Thus, resolving parallel to x, we have

$$\iiint \rho X dxdydz + \iint (lX_x + mX_y + nX_z)\, dS = 0 \ldots\ldots\ldots (2),$$

the volume-integration extending to all points within S, and the surface-integration to all points on S.

In transforming the left-hand side of (2) and similar expressions, we have to use a theorem of Integral Calculus discovered by Green, and expressible by means of the equation

$$\iiint \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) dxdydz = \iint (l\xi + m\eta + n\zeta)\, dS,$$

in which ξ, η, ζ are functions of x, y, z, which are finite, continuous, and one-valued within the surface S, and the volume-integration extends to all points within S, and the surface-integration to all points on S. This theorem we shall always refer to as *Green's Transformation.*

Applying the theorem to transform the surface-integral in (2) into a volume-integral, we obtain the equation

$$\iiint\left(\rho X+\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right)dx\,dy\,dz=0 \quad \ldots\ldots\ldots (3).$$

Since this equation holds whatever the surface S may be, provided only that it is closed and altogether within the solid, it holds when S is reduced to a point, and thus we have the differential equation of equilibrium

$$\rho X+\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0 \quad \ldots\ldots\ldots\ldots\ldots (4).$$

In like manner, resolving parallel to y and z, we obtain the equations

$$\left.\begin{aligned}\rho Y+\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}=0\\ \rho Z+\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (5).$$

In order that the part of the solid within S may be in equilibrium the sum of the moments of all the forces applied to this part about any axis must vanish. By taking moments about the axis of x we get the equation

$$\iiint\rho\,(yZ-zY)\,dx\,dy\,dz + \iint[y\,(lZ_x+mZ_y+nZ_z)-z\,(lY_x+mY_y+nY_z)]\,dS=0 \ldots (6).$$

The surface-integral becomes by Green's transformation

$$\iiint\left[y\left(\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right)-z\left(\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right)\right]dx\,dy\,dz - \iiint(Y_z-Z_y)\,dx\,dy\,dz.$$

Hence using equations (5), and reducing S to a point, we have

$$Y_z-Z_y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Similarly $$Z_x-X_z=0, \quad X_y-Y_x=0 \ldots\ldots\ldots\ldots\ldots (8).$$

The theorem expressed by these equations is due to Cauchy.

In future we shall denote the stress-components[1] as follows:

$$\left.\begin{aligned}X_x=P, \quad Y_y=Q, \quad Z_z=R,\\ Y_z=Z_y=S, \quad Z_x=X_z=T, \quad X_y=Y_x=U\end{aligned}\right\} \quad \ldots\ldots\ldots (9).$$

[1] For the symmetry see Note on Double Suffix Notations at the end of chapter III.

Then the equations of equilibrium become

$$\left.\begin{aligned} \frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}+\rho X=0 \\ \frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial S}{\partial z}+\rho Y=0 \\ \frac{\partial T}{\partial x}+\frac{\partial S}{\partial y}+\frac{\partial R}{\partial z}+\rho Z=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

In case the solid is in a state of small internal relative motion we have to put instead of ρX, $\rho\left(X-\frac{\partial^2 u}{\partial t^2}\right)$, and similar expressions in the other equations, where $\frac{\partial^2 u}{\partial t^2}$ is the acceleration parallel to the axis x of the element whose mass is $\rho dx dy dz$. Thus the equations of small motion are three such as

$$\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial T}{\partial z}+\rho X=\rho\frac{\partial^2 u}{\partial t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

To obtain the boundary-conditions that hold at the surface of the solid, we suppose that surface-tractions, whose components parallel to the axes are F, G, H per unit area, are applied over this surface. Then the equations that hold at any point of the surface are

$$\left.\begin{aligned} F=lP+mU+nT \\ G=lU+mQ+nS \\ H=lT+mS+nR \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

where l, m, n are the direction-cosines of the normal to the surface drawn outwards.

The linearity of these equations is important in the general theory. We may infer from (12) and (10) that, if forces be applied at the boundary, stresses will exist in the interior, and the expression of these stresses is linear in the expression of the bodily forces and surface-tractions.

15. Statement of the Assumptions made.

The establishment of Cauchy's theorem, and of the equations (4) and (5), rests on a particular assumption not quite fully stated in art. 13. The bodily force that acts upon any element is necessarily the product of a finite quantity and the mass of the element. From this equations (4) and (5) are deduced. But, in order that Cauchy's theorem may be true, it is necessary to assume

that the moments of the bodily forces that act upon any element, about some set of axes whose origin is in the element, are ultimately the products of the mass of the element and small quantities which vanish when the linear dimensions of the element vanish, in other words that the bodily forces acting upon any element are ultimately reducible to a single force in a line that meets the element. This is true for such bodily forces as gravitation, and we shall limit our enquiry to the cases for which it holds good. In the theory of Magnetism and some other parts of mathematical Physics it has to be supposed that the bodily forces applied to an element of the medium reduce to a force through the centre of the element and a couple[1]. When this is the case we may take L, M, N for the components of the couple, and then to equation (6) we have to add the term $\iiint L dx dy dz$, so that instead of (7) and (8) we get

$$\left.\begin{aligned} Z_y - Y_z + L &= 0 \\ X_z - Z_x + M &= 0 \\ Y_x - X_y + N &= 0 \end{aligned}\right\}.$$

In the theory of Elasticity these considerations are unimportant, and the equations (7) and (8) are true for the systems of forces usually considered.

16. Transformation of stress-systems.

We must now consider the transformation of any stress-system from one set of rectangular coordinates to another.

Denote the new system of coordinates by x', y', z', and the corresponding stresses by P', Q' R', S', T', U', and let the scheme of transformation be

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

.....(13).

[1] For a mechanical representation of a medium in which this would be the case, see Larmor 'On the propagation of a disturbance in a gyrostatically loaded medium'. *Proc. Lond. Math. Soc.* Nov. 1891.

The stress across the plane x', or $l_1x+m_1y+n_1z=0$, has components

$$l_1P+m_1U+n_1T,$$
$$l_1U+m_1Q+n_1S,$$
$$l_1T+m_1S+n_1R;$$

hence

$$P'=l_1(l_1P+m_1U+n_1T)+m_1(l_1U+m_1Q+n_1S)+n_1(l_1T+m_1S+n_1R),$$

and

$$U'=l_2(l_1P+m_1U+n_1T)+m_2(l_1U+m_1Q+n_1S)+n_2(l_1T+m_1S+n_1R);$$

or

$$\left.\begin{aligned} P'&=l_1^2P+m_1^2Q+n_1^2R+2m_1n_1S+2n_1l_1T+2l_1m_1U,\\ U'&=l_1l_2P+m_1m_2Q+n_1n_2R+(m_1n_2+m_2n_1)S+(n_1l_2+n_2l_1)T+(l_1m_2+l_2m_1)U \end{aligned}\right\} \quad \ldots\ldots(14),$$

and from these Q', R', and S', T' may be written down by cyclical interchanges of the suffixes 1, 2, 3.

It is to be observed that, if the quadric

$$(P, Q, R, S, T, U\chi xyz)^2=k \quad \ldots\ldots\ldots\ldots\ldots (15)$$

be transformed by the substitutions given by the scheme (13), it becomes

$$(P', Q', R', S', T', U'\chi x'y'z')^2=k.$$

This quadric is termed the *Stress-Quadric*, and it appears from the above equations that the normal stress across any plane is inversely as the square of the radius vector of the stress-quadric which is normal to the plane.

Again the transformation may be such as to refer the quadric to its principal planes. When this is so S', T', U' will vanish, and the stress across any one of the coordinate planes is normal to the plane. The stresses P', Q', R' are then called *principal stresses*, and we learn that there are at any point of the solid three co-orthogonal planes across each of which the stress is purely normal. These are called the *principal planes of stress*.

17. Shearing stress.

Let us consider a case of stress in two dimensions. Suppose that S, T, U all vanish, so that the axes of (x, y, z) are the principal

axes of the stress, and consider the stress across the planes $x = \pm y$, when P and Q are equal and of opposite sign. Then

$$l_1 = m_1 = \frac{1}{\sqrt{2}}, \quad n_1 = 0,$$

$$-l_2 = m_2 = \frac{1}{\sqrt{2}}, \quad n_2 = 0,$$

$$l_3 = m_3 = 0, \quad n_3 = 1.$$

Hence $P' = 0, \quad Q' = 0, \quad$ and $\quad U' = -P = Q,$

so that equal pressure and tension across two planes at right angles to each other are equivalent to tangential stresses across the planes bisecting them of amount equal to either; and, conversely, equal tangential stresses in any two planes at right angles are equivalent to two equal normal stresses on planes inclined at an angle of 45° to them, of which one is a pressure and the other a tension.

Thus the following are equivalent stress diagrams:

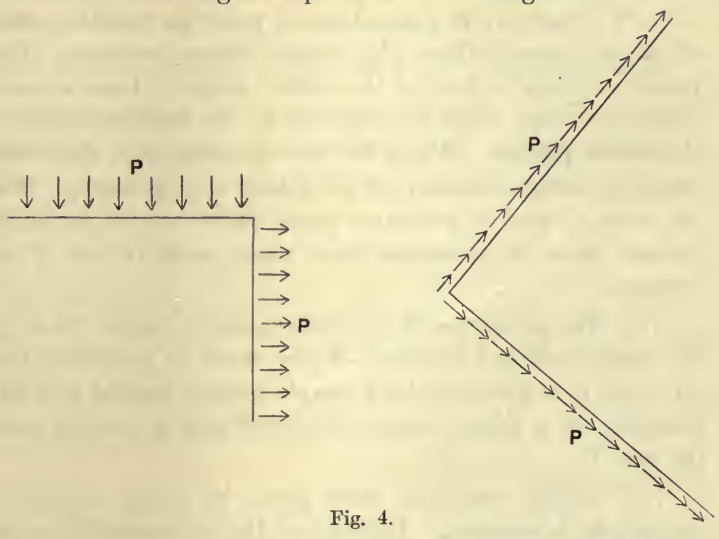

Fig. 4.

Such a stress-system may be expected to produce a shear of the planes across which it acts, and it is for this reason that tangential stress is called shearing stress[1]. Properly speaking, shearing stress consists of equal tangential stresses on two perpendicular planes, the directions of both being perpendicular to the intersection of the planes.

[1] Historically *shear* appears to have been first used by engineers to express what we here call *shearing stress*.

We shall leave to the reader the working out of the following:

(i) There are three invariants of the stress at any point, *i.e.* functions of the stresses which are independent of the rectangular axes of reference.

(ii) If P_1, Q_1, R_1 be the principal stresses, the stress across any plane is proportional to the perpendicular on the parallel tangent plane of the ellipsoid $x^2/P_1^2 + y^2/Q_1^2 + z^2/R_1^2 = \text{const.}$

This is Lamé's *stress-ellipsoid.* The reciprocal surface was considered by Cauchy, its radius vector in any direction is inversely proportional to the stress across the perpendicular plane.

(iii) The quadric $x^2/P_1 + y^2/Q_1 + z^2/R_1 = \text{const.}$, called Lamé's *stress-director quadric,* is the reciprocal of the stress-quadric with respect to its centre; it has the property that the direction of the stress across any plane is that of the radius-vector to the point of contact of the parallel tangent plane.

(iv) There are in general at any point an infinite number of planes across which the normal stress vanishes. These planes envelope a cone of the second degree, (Lamé's *cone of shearing stress,*) which is reciprocal to the asymptotic cone of the stress-quadric. When the cone is imaginary, the normal stress is either a tension for all planes or a pressure. When the cone is real it separates those planes across which the normal stress is a tension from those across which it is a pressure.

(v) The properties of the stress-quadric enable us to find the magnitude and direction of the stress in particular cases *e.g.* when it is known to be a simple tension parallel to a fixed direction, or a simple shearing stress of sets of parallel planes (cf. art. 7).

(vi) If the stress on every plane be purely normal, its magnitude is constant. In this case the stress-quadric, stress-ellipsoid, and stress-director quadric are spheres.

(vii) If two lines (1) and (2) be drawn from any point of a solid, and planes perpendicular to them be drawn at the point, then the component parallel to (2) of the stress across the plane perpendicular to (1) is equal to the component parallel to (1) of the stress across the plane perpendicular to (2). (*Cauchy's theorem.*) This is the generalisation of the theorem expressed by equations (7) and (8).

18. Work done in slightly increasing the strain.

We have seen that the strain in the neighbourhood of the point (x, y, z) of an elastic solid, which is slightly displaced so that its coordinates become $x+u, y+v, z+w$, can be expressed by means of six components of strain e, f, g, a, b, c viz.:

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial x}, & f &= \frac{\partial v}{\partial y}, & g &= \frac{\partial w}{\partial z} \\ a &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, & b &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, & c &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned}\right\} \ldots\ldots (16),$$

and we have seen that the stress can be expressed by means of six components of stress P, Q, R, S, T, U. We shall now investigate an expression for the work done in slightly increasing the strain.

Let F, G, H be the applied surface-tractions given by (12), and suppose the displacements slightly increased from u, v, w to $u+\delta u$, $v+\delta v$, $w+\delta w$, then the work done is

$$\iint (F\delta u + G\delta v + H\delta w)\, dS' + \iiint \rho\, (X\delta u + Y\delta v + Z\delta w)\, dxdydz \ldots (17),$$

the surface-integration extending to all points of the surface S' of the solid, and the volume-integration to all internal points. Substituting for F, G, H from (12), and using Green's transformation, we find that (17) becomes

$$\begin{aligned} \iiint \Big[& P\frac{\partial \delta u}{\partial x} + Q\frac{\partial \delta v}{\partial y} + R\frac{\partial \delta w}{\partial z} + S\left(\frac{\partial \delta w}{\partial y} + \frac{\partial \delta v}{\partial z}\right) \\ & + T\left(\frac{\partial \delta u}{\partial z} + \frac{\partial \delta w}{\partial x}\right) + U\left(\frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial y}\right) \Big]\, dxdydz \\ + \iiint \Big[& \left(\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} + \rho X\right)\delta u + \left(\frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} + \rho Y\right)\delta v \\ & + \left(\frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} + \rho Z\right)\delta w \Big]\, dxdydz. \end{aligned}$$

The latter of these integrals vanishes identically in virtue of the equations of equilibrium, and therefore the work done in increasing the strain from e, f, g, a, b, c to $e+\delta e, f+\delta f, \ldots c+\delta c$ is

$$\iiint (P\delta e + Q\delta f + R\delta g + S\delta a + T\delta b + U\delta c)\, dxdydz \ldots\ldots (18).$$

Thus the increment of the potential energy per unit volume stored up in the solid by the strain is

$$P\delta e + Q\delta f + R\delta g + S\delta a + T\delta b + U\delta c \ldots\ldots\ldots\ldots (19).$$

If there exist a function W of e, f, g, a, b, c of which this is the total differential, then

$$\left.\begin{aligned} P=\frac{\partial W}{\partial e}, \quad Q=\frac{\partial W}{\partial f}, \quad R=\frac{\partial W}{\partial g} \\ S=\frac{\partial W}{\partial a}, \quad T=\frac{\partial W}{\partial b}, \quad U=\frac{\partial W}{\partial c} \end{aligned}\right\} \ldots\ldots\ldots\ldots (20),$$

and W is the potential energy of the strained elastic solid in the condition denoted by (e, f, g, a, b, c). We shall shew in chapter V. that W exists when the solid is strained at constant temperature, or is executing small vibrations.

19. The measurement of stress.

We have seen that any stress-system can be expressed by means of six component stresses. These are forces per unit area, and the dimension symbol of a stress is therefore $ML^{-1}T^{-2}$.

Stresses may be measured in dynes per square centimetre, or in poundals per square foot, or generally in units of force per unit area. A unit stress is a stress of one unit of force per unit area. Thus if one foot be the unit of length, one second the unit of time, and one lb. the unit of mass, the pressure of the atmosphere (15 lbs. weight on the square inch) is $144 \times 15 \times 32{\cdot}2$ units of stress.

In the theory of elastic solids much larger stresses are considered. Stresses of thousands of lbs. weight per square inch have to be allowed for in calculations made by engineers. Prof. Unwin cites as an example the Conway Bridge, which is daily subjected to stresses reaching "7 tons per square inch". Ordinary railway axles are expected to be quite safe with a stress of "4 tons to the inch".

20. Stress in a medium.

We have defined stress in terms of intermolecular force, assuming the molecular structure of matter, but we may generalise the notion thus defined so as to get rid of the molecular hypothesis. Let us consider space, or any portion of it, continuously filled with a medium, whose parts move in obedience to Newton's Laws of Motion. Then in such a medium the change of momentum of any part will be partly produced by the action upon it of the surrounding parts. This action between the two parts separated by any surface can be conceived of as exerted across the surface,

and in that case must reduce to stress-systems such as we have been investigating. We must therefore suppose that at every point in a medium, across any small area containing the point, there exists a pair of equal and opposite forces proportional to the area. The existence of this stress might be taken as a fundamental fact, just as in Elementary Mechanics the tension of a string and the pressure of a fluid are taken as fundamental notions derived from experience, and we may use the notion to obtain equations of motion or equilibrium, without seeking to refer it to some more or less remote cause. As an example of a doctrine of stress derived from an hypothesis other than molecular we may cite the theory of Hydrodynamics. It can be shewn that the differential equation of surfaces which always contain the same particles has three independent integrals, so that there are three families of such surfaces, and the individuality of any portion of the medium is therefore secured by the kinematical conditions. Now, assuming that no part of the medium can be created or destroyed, we have an equation which represents the fact that the increase in any time of the amount of matter within any closed surface is equal to the amount that has entered the surface from outside during the time, less the amount that has left it from inside. If we form equations of motion, assuming this kinematical condition after the manner of Lagrange, it appears that the pressure of the fluid is the force called into play to prevent the violation of the kinematical condition. We can thus see how the notion of hydrostatic pressure can be arrived at kinematically without the intervention of any molecular hypothesis. The stress-systems we have been investigating would result from generalising the notion thus obtained, by assuming that, when any relative motions of the parts of a continuous medium take place, forces are called into play between the parts of the medium analogous to hydrostatic pressure, but not necessarily normal to the separating surfaces.

CHAPTER III.

ELASTIC PROPERTIES OF MATTER—STRESS-STRAIN RELATIONS[1].

21. When a solid body is subjected to the action of a system of forces, which, acting on a rigid body, would produce equilibrium, certain effects are observable depending on the nature and magnitude of the applied forces, and from the observed effects certain inferences can be drawn as to the behaviour of the system of molecules composing the body. The observable effects are changes of temperature and deformations of the body, and the latter can be kinematically expressed by the strain-systems previously investigated. The applied forces (*load*) are in general of the character of surface-tractions and bodily forces, and we have seen that the existence of stresses within the solid can be inferred. The expression of these stresses is linear in the expressions of the

[1] The following among other authorities may be consulted:

Sir G. Stokes. 'On the theories of...the Equilibrium and Motion of Elastic Solids'. *Camb. Phil. Soc. Trans.* VIII. 1845, and *Mathematical and Physical Papers*, vol. I.

Sir W. Thomson. Art. *Elasticity* in *Encyclopædia Britannica* and *Mathematical and Physical Papers*, vol. III. Also, *Lectures on Molecular Dynamics*. Baltimore, 1884.

F. E. Neumann. *Vorlesungen über die Theorie der Elasticität der festen Körper und des Lichtäthers*, 1885.

Saint-Venant. Translation of Clebsch's *Théorie de l'élasticité des corps solides*. Paris, 1883.

Voigt. 'Allgemeine Formeln für die Bestimmung der Elasticitätsconstanten von Krystallen ..', Wiedemann's *Annalen*, XVI. 1882, 'Bestimmung der Elasticitätsconstanten von Beryll und Bergkrystall', Wiedemann's *Annalen*, XXXI. 1887, 'Bestimmung der Elasticitätsconstanten von Topas und Baryt', Wiedemann's *Annalen*, XXXIV. 1888, 'Bestimmung der Elasticitätsconstanten von Fluss-spath, Pyrit, Steinsalz, Sylvin', Wiedemann's *Annalen*, XXXV. 1888, and 'Ueber die Beziehung zwischen den beiden Elasticitätsconstanten isotroper Körper', Wiedemann's *Annalen*, XXXVIII. 1889.

applied forces and surface-tractions, or more generally the internal stress is linear in terms of the load. When we consider the solid as composed of molecules, the stresses are seen to arise from variations in the forces exerted between the molecules, depending, partly at least, on variations in the molecular configuration; and we have accordingly included in our statement the inference that the application of force to the solid is accompanied by changes in its molecular configuration. Such changes may be of two kinds. Either the new configuration is a configuration of equilibrium without the applied force, or it is not; and, if it be not a configuration of equilibrium, yet it may still be nearer to some other configuration of equilibrium than to the original configuration.

22. Definitions.

Unless every new configuration is a configuration of equilibrium with no applied force the solid is said to be *elastic*. In other words a body is elastic if a continued application of force is required to maintain any alteration of volume or figure.

The term *strain* connotes all the relative displacements of the parts of the solid.

When the solid is elastic, part of the strain disappears on the removal of the load. This part is called *elastic strain* and the part which does not disappear is called *set*.

Set is a permanent rearrangement of the parts of the solid.

There are in general limits to the amount of strain of any kind which a solid can undergo without taking any set. Such limits are called *elastic limits*. A solid, which can be strained without taking any set, is said to be in a *state of ease* up to the amount of the strain at which set begins. An elastic solid is said to be *perfectly elastic* within its elastic limits.

When a perfectly elastic solid is strained by the application of a load, which is not great enough to produce a strain surpassing the elastic limit, the whole strain disappears when the load is removed.

23. Hooke's Law.

This law connects the observed deformation of an elastic solid body with the applied load. It states that

The deformation produced is proportional to the load producing it.

We have already noted the linearity of the relations between internal stress and load, we may therefore restate the law in the form

Stress is a linear function of strain.

This is the *generalised Hooke's law* to which we have referred. It forms the foundation of the Mathematical Theory of Elasticity as at present developed.

Hooke's law applies, in the first instance, to an elastic solid body in a state of ease, strained within its elastic limits.

If the law be true it may be applied to finite strains not exceeding the amount indicated, we have therefore to consider in the first place the proofs of the law.

The proof of the law in the cases in which it holds is experimental. The law has been demonstrated by the most minute and careful experiments to hold for nearly all elastic solid bodies in common use. Sir William Thomson states that the "law is fulfilled, as accurately as any experiments hitherto made can tell, for all metals and hard solids, each through the whole range within its limits of elasticity". Experimenters have however noted a defect of the law in the case of cast-iron and other cast metals for which it appears that the law does not represent the stress-strain relation for any strains large enough to be observed.

A simple experimental proof can be given in all cases, when the strain is infinitesimal. The strain is in this case expressed by linear functions of the differential coefficients of the displacements. If the law be true, the differential equations of vibration of an elastic solid body become linear, and isochronous vibrations will be possible. The experimental fact, that all solids can be thrown into a state of isochronous vibration, is a peremptory proof of the truth of the law in the case of the very small displacements involved. This was first pointed out by Sir G. Stokes.

When a solid is strained by unequal heating it is clear that Hooke's Law as formulated above does not hold, for a strain is produced without any load producing it. The above discussion must be regarded as applying to solids strained at constant temperature by the application of external forces, and to such strains as take place in sound vibrations. In case the strain is not of this kind there exists no theory which can be demonstrated in a thoroughly satisfactory manner.

24. Isotropy—Æolotropy.

A solid is said to be elastically *isotropic* when it exhibits no differences of elastic quality depending on direction. The same forces, applied in the same way, to similar and similarly situated portions of two equal spherical or cubical blocks cut in any manner from an isotropic solid, produce in them the same strain.

Elastic solids which are not isotropic are termed *œolotropic*. If two spherical portions be cut in the same manner from an œolotropic solid, and one of them be turned through any angle about any axis, and thereafter similar systems of forces be similarly applied to similar and similarly situated portions of them, the strains produced will be in general different. Or if two cubical blocks be cut from the solid, whose edges are not initially parallel, and similar systems of forces be similarly applied to them, different systems of strains will be produced in the two blocks. In fact the resistance, which an œolotropic solid offers to change of shape, depends partly on the load, and partly on the *direction*, with reference to axes fixed in the solid, of the part of the surface to which it is applied. As an example of the difference between isotropy and œolotropy, we have the well-known result that a weight, hung to the end of an isotropic bar, simply extends the bar, while, if the bar be œolotropic, it may be twisted as well as extended.

25. Elastic constants—Moduluses.

According to the generalised Hooke's Law the stress at any point is a linear function of the strain, *i.e.* each of the six stresses P, Q, R, S, T, U is a linear function of the six strains e, f, g, a, b, c.

The coefficients in the expression of $P\ldots$ as functions of $e\ldots$ are termed *elastic constants*.

A *Modulus of Elasticity* is the quantity obtained by dividing the number expressing a stress by the number expressing the corresponding strain[1]. The moduluses are functions of the elastic constants.

A Modulus is called a *principal modulus* when the stress concerned provokes only a strain of its own type. In general this is not the case, but a stress of given type being applied to a solid,

[1] *e.g.* P and e are corresponding stresses and strains; so S and a, and the average pressure $-\frac{1}{3}(P+Q+R)$ and the cubical compression $-(e+f+g)$.

strains of the corresponding type and of other types will be found. A good example is that of extension. When a prismatic bar is extended, the stress at any point can reduce to normal tension across the sections of the bar, but the strain will not at the same time reduce to simple extension of fibres of the bar parallel to its axis. The extension is in fact accompanied by lateral contractions. For the definition of a modulus of any kind, we have to suppose that only a certain kind of stress exists, and then calculate the amount of the strain of the corresponding type, that must occur. Typical examples of these calculations will be found in arts. 41–43 below.

The moduluses and elastic constants of a solid are physical quantities of the same kind as stress, *i.e.* they are forces per unit area. In many treatises and memoirs, constants are introduced which are the reciprocals of the moduluses and elastic constants, *e.g.* the "coefficient of compressibility" is the reciprocal of the modulus of compression.

26. Constants of Isotropic solid.

To give a simple theory of homogeneous isotropic solids, we may assume that a uniform hydrostatic pressure, or tension, applied to the surface of a cubical block, produces uniform cubical compression, or dilatation, at all points of the interior and no other strain.

We may also assume that uniform tangential stress, applied to any plane, produces uniform shear of the plane and no other strain.

The *modulus of compression*, or *bulk-modulus*, of an isotropic solid is the quantity obtained by dividing the uniform tension by the uniform dilatation produced by it. It will be denoted by k.

The *modulus of rigidity* is the quantity obtained by dividing the uniform tangential stress by the shear produced by it. It will be denoted by μ. These are principal moduli.

Suppose the uniform tension to be p, and the small uniform extension of all lines of the material to be e, so that a cube whose sides are each 1 becomes a cube whose sides are each $1+e$, then the dilatation produced is $3e$, and we have the equation

$$p = 3ke \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again suppose the tangential stress, of amount S, to be applied

to the plane (yz) in a direction parallel to y, the shear of the plane will be a, where

$$S = \mu a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Young's Modulus E is the ratio of a tractive force P per unit area, applied to a bar of the material, to the extension e produced in the bar, so that $P = Ee$.

We shall now obtain the expression of the six stresses in terms of the six strains, and find the value of E as a function of k and μ.

27. Stress-strain relations for Isotropic solid.

A method of arriving at the theory is founded on the following assumption:

In the case of an isotropic solid, the principal planes of stress are perpendicular to the principal axes of extension.

Suppose the stress and strain referred to principal axes. Let P_1, Q_1, R_1 be the components of stress, and e_1, f_1, g_1 the components of strain. Then plainly P_1 is a symmetrical function of f_1 and g_1; and, since the stresses are linear functions of the strains, we may write

$$P_1 = \lambda(e_1 + f_1 + g_1) + 2\mu' e_1,$$
$$Q_1 = \lambda(e_1 + f_1 + g_1) + 2\mu' f_1,$$
$$R_1 = \lambda(e_1 + f_1 + g_1) + 2\mu' g_1,$$

the coefficients λ and μ' being the same in all three equations, as there is no difference of elastic quality depending on direction.

Now let $P_1 = Q_1 = R_1$, then will $e_1 = f_1 = g_1$, and we find by (1)

$$3\lambda + 2\mu' = 3k.$$

Next let $P_1 = -Q_1$, and $R_1 = 0$. Then P_1 and Q_1 are equivalent to a shearing stress of the planes bisecting the angles between the principal planes, across which P_1 and Q_1 act. The magnitude of the shearing stress is P_1, and the shear produced is P_1/μ, and this is equal to $2e_1$,[1] since in the case supposed $e_1 = -f_1$, and $g_1 = 0$. Hence $\mu' = \mu$, so that we have

$$\lambda = k - \tfrac{2}{3}\mu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It is convenient to use λ and μ as the fundamental elastic constants of an isotropic solid, and then the stress-strain relations, referred to principal axes, are three such as

$$P_1 = \lambda(e_1 + f_1 + g_1) + 2\mu e_1 \ldots\ldots\ldots\ldots\ldots (4).$$

[1] See arts. 11 and 17.

Now transform this stress-strain system to any rectangular axes of (x', y', z'). Let (x, y, z) be the coordinates of (x', y', z') referred to the principal axes, and let the scheme of transformation be

·	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
	l_3	m_3	n_3

The stress-quadric $(P, Q, R, S, T, U)(x'y'z')^2 = \text{const.}$ is the transformed of $P_1x^2 + Q_1y^2 + R_1z^2 = \text{const.}$, and the elongation-quadric $(e, f, g, \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)(x'y'z')^2 = \text{const.}$ is the transformed of $e_1x^2 + f_1y^2 + g_1z^2 = \text{const.}$, and equations (33) of art. 11, and (14) of art. 16, give us

$$\begin{aligned} P &= P_1l_1^2 + Q_1m_1^2 + R_1n_1^2 \\ &= \lambda(e_1 + f_1 + g_1) + 2\mu(l_1^2e_1 + m_1^2f_1 + n_1^2g_1) \\ &= \lambda(e + f + g) + 2\mu e, \end{aligned}$$

and

$$\begin{aligned} U &= l_1l_2P_1 + m_1m_2Q_1 + n_1n_2R_1 \\ &= (l_1l_2 + m_1m_2 + n_1n_2)\lambda(e_1 + f_1 + g_1) + 2\mu(l_1l_2e_1 + m_1m_2f_1 + n_1n_2g_1) \\ &= \mu c. \end{aligned}$$

Hence the expressions for the six stresses are in general

$$\left.\begin{aligned} P &= \lambda\Delta + 2\mu e \\ Q &= \lambda\Delta + 2\mu f \\ R &= \lambda\Delta + 2\mu g \\ S &= \mu a \\ T &= \mu b \\ U &= \mu c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where

$$\Delta = e + f + g \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

is the cubical dilatation.

With the above expressions for the stresses in terms of the strains it is found that the expression

$$Pde + Qdf + Rdg + Sda + Tdb + Udc$$

is the complete differential of a function W of e, f, g, a, b, c, given by the equation

$$2W = (\lambda + 2\mu)(e+f+g)^2 + \mu(a^2+b^2+c^2-4fg-4ge-4ef) \ldots (7),$$

so that the stresses $P, Q, \ldots U$ are given by the equations

$$P = \frac{\partial W}{\partial e}, \quad Q = \frac{\partial W}{\partial f}, \ldots U = \frac{\partial W}{\partial c} \ldots\ldots\ldots\ldots\ldots (8).$$

28. Relations between elastic constants.

To express E in terms of λ and μ, suppose the stresses reduce to a simple tension P. Then a, b, c will be zero, and

$$(\lambda+2\mu)e + \lambda(f+g) = P,$$
$$(\lambda+2\mu)f + \lambda(g+e) = 0,$$
$$(\lambda+2\mu)g + \lambda(e+f) = 0,$$

from which $$f = g = -\frac{\lambda}{2(\lambda+\mu)}e = -\sigma e \text{ say,}$$

and $$P = \left[\lambda + 2\mu - \frac{\lambda^2}{(\lambda+\mu)}\right]e = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}e.$$

Hence Young's Modulus is E, where

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{9\mu k}{3k+\mu} \ldots\ldots\ldots\ldots (9).$$

The number $$\sigma = \tfrac{1}{2}\lambda/(\lambda+\mu) \ldots\ldots\ldots\ldots (10)$$

is the *ratio of lateral contraction to longitudinal extension* when a bar is pulled out. This constant is called *Poisson's ratio.* According to the molecular hypothesis of Cauchy and Poisson it is equal to $\frac{1}{4}$, and λ is equal to μ. It is certain that there are materials for which experiment shews that λ is at any rate very nearly equal to μ. We shall not however introduce the relation $\lambda = \mu$, except occasionally in numerical calculations.

In general the ratio σ must lie between $\frac{1}{2}$ and -1; for if $\sigma > \frac{1}{2}$, then μ is negative, or the medium would not resist distortion, and if $-1 > \sigma$, $k\,(=\lambda + \frac{2}{3}\mu)$ is negative and the medium would not resist compression. These limits for σ are theoretically necessary. As a matter of practice there are no known isotropic materials for which σ is negative, and a negative value of σ is for physical reasons highly improbable.

We have introduced 5 constants $E, \sigma, \lambda, \mu, k$ of which only two are independent; the reader will find it useful to make for himself a table giving expressions for each of these in terms of any selected two.

29. Equations of Equilibrium and small Motion.

Let u, v, w be the component displacements of any point (x, y, z) of the body, and suppose them small continuous functions of x, y, z. Then the strain-components e, f, g, a, b, c are given by equations (36) of art. 12, so that equations (5) become such equations as

$$\left.\begin{aligned} P &= \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x} \\ S &= \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (11).$$

Substitute these in the equations of small motion (11) of art. 14, and we have

$$\left.\begin{aligned} (\lambda+\mu)\frac{\partial \Delta}{\partial x}+\mu\nabla^2 u+\rho X &= \rho\frac{\partial^2 u}{\partial t^2} \\ (\lambda+\mu)\frac{\partial \Delta}{\partial y}+\mu\nabla^2 v+\rho Y &= \rho\frac{\partial^2 v}{\partial t^2} \\ (\lambda+\mu)\frac{\partial \Delta}{\partial z}+\mu\nabla^2 w+\rho Z &= \rho\frac{\partial^2 w}{\partial t^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12),$$

where ∇^2 denotes the operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, Δ is the cubical dilatation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$, ρ is the density, and X, Y, Z are the components of the bodily forces per unit mass acting at the point (x, y, z) of the body. The equations of equilibrium are the same as (12) with the right-hand sides put equal to zero.

With the notation of art. 12 for the cubical dilatation and the three rotations, these equations can also be written in the form

$$\left.\begin{aligned} (\lambda+2\mu)\frac{\partial \Delta}{\partial x}-2\mu\frac{\partial \varpi_3}{\partial y}+2\mu\frac{\partial \varpi_2}{\partial z}+\rho X &= \rho\frac{\partial^2 u}{\partial t^2} \\ (\lambda+2\mu)\frac{\partial \Delta}{\partial y}-2\mu\frac{\partial \varpi_1}{\partial z}+2\mu\frac{\partial \varpi_3}{\partial x}+\rho Y &= \rho\frac{\partial^2 v}{\partial t^2} \\ (\lambda+2\mu)\frac{\partial \Delta}{\partial z}-2\mu\frac{\partial \varpi_2}{\partial x}+2\mu\frac{\partial \varpi_1}{\partial y}+\rho Z &= \rho\frac{\partial^2 w}{\partial t^2} \end{aligned}\right\} \quad \ldots\ldots (13).$$

The boundary-conditions are found from equations (12) of art. 14, by inserting the expressions for the six stresses in terms of the six strains. Thus we obtain

$$\left.\begin{aligned} F &= l(\lambda\Delta+2\mu e)+m\mu c+n\mu b \\ G &= l\mu c+m(\lambda\Delta+2\mu f)+n\mu a \\ H &= l\mu b+m\mu a+n(\lambda\Delta+2\mu g) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (14).$$

It is easy to shew that these can be written in the form

$$\left.\begin{aligned} F &= l\lambda\Delta + 2\mu\left(\frac{\partial u}{\partial n'} + m\varpi_3 - n\varpi_2\right) \\ G &= m\lambda\Delta + 2\mu\left(\frac{\partial v}{\partial n'} + n\varpi_1 - l\varpi_3\right) \\ H &= n\lambda\Delta + 2\mu\left(\frac{\partial w}{\partial n'} + l\varpi_2 - m\varpi_1\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots (15),$$

where l, m, n are the direction-cosines of the normal to the bounding surface drawn outwards, dn' is the element of this normal, and F, G, H are the forces per unit area parallel to the axes applied at any point of the bounding surface.

30. Table of Elastic Constants.

As shewing the order of magnitude of the moduluses of some well-known elastic materials, and for convenience of reference, we give the following table.

Material	ρ	E	k	μ	σ	Authority
Steel Pianoforte Wire[1]	7·727	2049	.....	.....	.....	D. M'Farlane, quoted by Sir W. Thomson, *Encyc. Brit.*
Steel	7·849	2181	1876	834	[·306]	Everett[2]
„	.....	.....	.....	.....	·294	Kirchhoff, Pogg. *Ann.*, 1859
„	.....	2081	1499	.....	·268	Amagat, *Journal de Physique*, 1889
Iron (wrought)	7·677	2000	1485	785	[·274]	Everett
Brass (drawn)	8·471	1096	.....	373	[·468]	Everett
Brass	.....	.....	1063	.....	·333	Wertheim, *Ann. de Chimie*, 1848
„	.....	1106	1071	.....	·327	Amagat, *loc. cit.*
Copper	.....	1239	1189	[470]	·327	Amagat, *loc. cit.*
„	8·843	1258	1717	456	[·378]	Everett
Lead	.....	160	369	.....	·428	Amagat, *loc. cit.*
Glass	2·942	615	423	243	.....	Everett
„	.....	690	463	[253]	·245	Amagat, *loc. cit.*

For greater completeness, the density ρ of the matter referred to is also generally given. The moduluses are given in terms of a unit stress of 10^6 Grammes' weight per square centimetre The authority for the results is also given. It will be noticed

[1] Except in Amagat's experiments the materials tested were probably not isotropic, but they are treated as such by the authorities referred to.

[2] 'Units and Physical Constants'.

that in some cases we give E, in others k, in others μ, and in others σ. The numbers in square brackets occasionally given in some columns are obtained from those given in other columns by application of known relations between elastic constants.

31. Elastic Constants of Æolotropic Solid.

In the general case of an æolotropic solid subject to Hooke's Law we must have each of the six stresses a linear function of the six strains. Adopting a notation similar to that of art. 10, we may express the most general stress-strain relations in the form

$$(P, Q, R, S, T, U)$$

$$= \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{pmatrix} (e, f, g, a, b, c) \quad \ldots(16).$$

The quantities $c_{11}\ldots$[1] are called elastic constants.

We shall hereafter prove (chapter V.), that in case the solid is strained, either at constant temperature, or in such a way that no heat is gained or lost by any element, the work (per unit volume) done in slightly increasing the strain, expressed by

$$Pde + Qdf + Rdg + Sda + Tdb + Udc \ldots\ldots\ldots\ldots (17)$$

is the complete differential of a function W of the six strains e, f, g, a, b, c. In consequence of equations (16), this function must be a complete quadratic function of the six strains, and this function is

$$\tfrac{1}{2}(c_{11}, c_{22}, \ldots c_{66}, c_{12}\ldots)(e, f, g, a, b, c)^2 \ldots\ldots\ldots\ldots (18),$$

where

$$c_{rs} = c_{sr}, \ (r, s = 1, 2 \ldots 6) \ldots\ldots\ldots\ldots\ldots (19).$$

There are 15 relations of the form (19), whereby the 36 constants of equation (16) are reduced to 21. This is Green's reduction of the number of constants, and, in developing the theory, we shall suppose it to hold good. The constants $c_{11}\ldots$ for any particular solid depend on the material, and on the directions chosen as axes.

[1] For the symmetry see Note on Double Suffix Notations at the end of this chapter.

Again we shall prove that, for a system of discrete particles homogeneously arranged, whose action upon each other is such that the force between any two is in the line joining them and is a function of their distance, there is a further reduction in the number of constants from 21 to 15, effected by the equations

$$\left.\begin{aligned} c_{23} = c_{44}, \quad c_{31} = c_{55}, \quad c_{12} = c_{66} \\ c_{14} = c_{56}, \quad c_{25} = c_{64}, \quad c_{36} = c_{45} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (20).$$

This is Cauchy's reduction of the number of constants, and there are many practical applications for which it is convenient to suppose it to hold good. We shall not however expressly introduce it into the general theory, as we have no sufficient ground for supposing that it expresses a necessary property of solid bodies, and it has not been verified by experiment.

32. Natural Crystals.

Further reductions in the number of constants take place, when the solid exhibits any kind of structural symmetry. The theory of the possible symmetries, and of the forms of bodies possessing these symmetries, is the science of Crystallography[1]. The internal structure of crystalline bodies can be inferred from the forms in which they crystallize. These forms are always bounded by surfaces which are very nearly plane, and deviations from plane forms are treated as exceptions, crystals bounded by planes being regarded as the rule. If any three edges of a crystal be chosen as axes, the planes of the crystal may be referred to them, and any plane can be determined by its intercepts on the axes. The *law of crystal form* experimentally ascertained is that, for any crystal, these intercepts are rational numerical multiples of three fixed lengths dependent on the form. The ratios of these fixed lengths are in general irrational, and are called *axial ratios*. Crystals are classified according to their symmetry, and their axial ratios, when referred to the most convenient system of axes. Thus if a, b, c be three fixed intercepts on three fixed axes, any crystal must be bounded by planes parallel to the planes whose intercepts are $a/h, b/k, c/l$, where h, k, l are positive or negative integral numbers. The ratios $a : b : c$ are the axial ratios, and depend on the material. The symbol (h, k, l)

[1] The system of notation adopted is that of Miller, see *e.g.*, G. H. Williams, *Elements of Crystallography*. Macmillan, London, 1890.

represents a family of parallel planes. If the crystal possess symmetry with respect to the plane of two of the axes, (say the b-axis and the c-axis), then the existence of a plane face, forming one member of the family (h, k, l) requires the existence of a face forming one member of the family $(-h, k, l)$. The collection of all the planes required by this law forms a *complete* or *holohedral* crystal *form*. Of equal importance are the *partial* crystal forms arrived at by the selection of certain planes from those of any complete crystal form. If half the planes be selected the resulting form is said to be *hemihedral*, if one quarter *tetartohedral;* the half or quarter selected must however be chosen according to certain rules, depending on the symmetry of the crystal. If, when the axes are suitably chosen, any one of the axial ratios become rational, it is clear that this ratio may be taken to be unity, and the two axes concerned are said to be equivalent; if further these axes be normal to planes of symmetry, they are said to be *equivalent axes of symmetry*. The law of selection of planes to make a hemihedral form is that only such planes can occur as intersect equivalent axes of symmetry at the same distance from the origin, at the same inclination, and in equal numbers[1]. The selection of half the planes of a complete crystal form may either include or exclude pairs of parallel planes; in the former case the resulting form is said to be *hemihedral with parallel faces*, in the latter *hemihedral with inclined faces*.

In the theory of elastic crystals, it is convenient to introduce two sets of rectangular axes. The axes of (x, y, z) are perfectly general, and the axes of (x_0, y_0, z_0) are parallel to lines to which it is convenient to refer the faces of the crystal, (sometimes, but not always, crystallographic axes). We shall denote the displacements, stresses, and strains, referred to the latter system, by (u_0, v_0, w_0), $(P_0, Q_0, R_0, S_0, T_0, U_0)$, and $(e_0, f_0, g_0, a_0, b_0, c_0)$; and the most general system of elastic constants corresponding to (16), when referred to the axes of (x_0, y_0, z_0), will be denoted by a's with double suffixes instead of c's.

[1] An example will make this clearer. If no two axes of symmetry be equivalent, but three planes of symmetry be present, as in the rhombic system, a complete form is the octahedron $(\pm 1, \pm 1, \pm 1)$. A possible method of hemihedrism is by selection of the planes $(\pm 1, \pm 1, +1)$. If there be three equivalent axes of symmetry at right angles, as in the cubic system, this is not a possible method of hemihedrism.

Complete crystal forms are divided into six systems according to their symmetries. We shall exhibit the stress-strain relations for each of these systems, and for the most important related hemihedral forms, with reference to specially selected sets of axes of (x_0, y_0, z_0).

To do this we shall follow F. E. Neumann in his assumptions that *crystallographic symmetry is identical with symmetry in elastic quality, and the directions of equivalent axes of symmetry are elastically interchangeable.*

33. Triclinic, Anorthic, or Doubly-Oblique System—(21 Constants).

In this system there is no plane of symmetry, and no reduction takes place in the number of constants. The relations (16) with $c_{rs} = c_{sr}$ are the stress-strain relations.

34. Monoclinic or Oblique System—(13 Constants).

This system possesses one plane of symmetry. Let this be the plane (x_0, y_0), then P_0, Q_0, R_0, U_0 must remain unaltered, and the other stresses must change sign, when z_0 and w_0 are changed into $-z_0$ and $-w_0$, *i.e.* when a_0 and b_0 are changed into $-a_0$ and $-b_0$. Hence the coefficients a_{14}, a_{15}, a_{24}, a_{25}, a_{34}, a_{35}, a_{64}, a_{65} vanish, and the energy-function W is given by

$$2W = (a_{11}, a_{22}, a_{33}, a_{66}, a_{12} \ldots \chi e_0, f_0, g_0, c_0)^2 + (a_{44}, a_{55}, a_{45} \chi a_0, b_0)^2 \quad (20),$$

i.e. $2W$ consists of a complete quadratic function of e_0, f_0, g_0, c_0, and a complete quadratic function of a_0, b_0.

Let (h, k, l) denote any plane of a complete form of this system, referred to the axes of (x_0, y_0, z_0), then $(h, k, -l)$ must be a plane of the form, and, taking the two parallel planes $(-h, -k, -l)$ and $(-h, -k, l)$, we get the complete crystal form. These planes do not form the boundary of a crystal since they do not enclose a space. The faces of a crystal are generally the sets of planes belonging to several complete or partial *forms* of the same system.

The parallel-faced hemihedral forms would consist of the planes (h, k, l) and $(-h, -k, -l)$, or of the other pair. Each of these is identical with a complete triclinic form, and may therefore be rejected from our enumeration. If there were true monoclinic crystals exhibiting this mode of hemihedrism we could have the phenomenon of the combination of an apparently monoclinic form with an apparently triclinic, which has never been observed. We

shall in like manner reject all partial forms arrived at geometrically, which are identical with forms belonging to a different system of crystals.

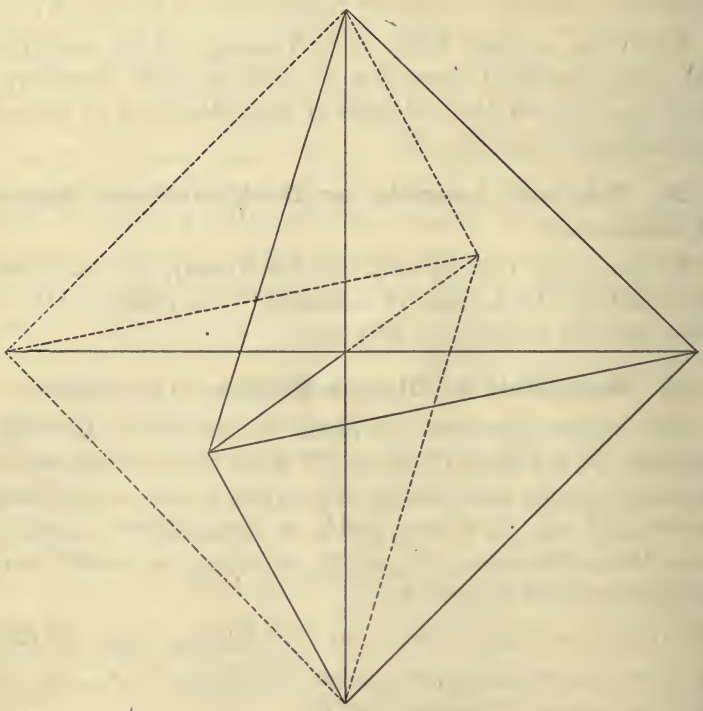

Fig. 5.

The inclined-faced hemihedral forms would consist of the planes (h, k, l) and $(-h, -k, l)$ or of the other pair, or again of the pair $(h, k, l,)$ and $(h, k, -l)$ or of the other pair. The first named have no plane of symmetry.

Some inclined-faced hemihedral forms of this system possess no plane of symmetry, but the figure of any such form will be similarly situated with respect to the axes if it be rotated through two right angles about the z_0 axis. It follows that P_0, Q_0, R_0, U_0 remain unaltered, and S_0, T_0 change sign when u_0, v_0, x_0, y_0 are changed into $-u_0, -v_0, -x_0, -y_0$ while w_0 and z_0 remain unaltered, *i.e.* when a_0 and b_0 are changed to $-a_0$ and $-b_0$. Hence the stress-strain formulæ for these are the same as for the complete forms of the same system. The remaining inclined-faced hemihedral forms of

this system possess one plane of symmetry, so that the formula (20) holds for all forms of this system.

35. Rhombic or Prismatic System—(9 Constants).

The complete forms of this system possess three planes of symmetry at right angles to each other. Let the planes (x_0, y_0) and (x_0, z_0) be planes of symmetry. Then all the coefficients $a_{14}, a_{15}, a_{16}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}, a_{36}, a_{45}, a_{46}, a_{56}$ vanish, and the energy-function is given by

$$2W = (a_{11}, a_{22}, a_{33}, a_{23}, a_{31}, a_{12}\chi e_0, f_0, g_0)^2 + a_{44}a_0^2 + a_{55}b_0^2 + a_{66}c_0^2 \quad (21),$$

which is the same as when there are three planes of symmetry. Topaz and Barytes are examples of crystals for which formula (21) holds.

Let (h, k, l) denote any plane of a complete form of this system referred to the axes (x_0, y_0, z_0), then $(\pm h, \pm k, \pm l)$ must all be

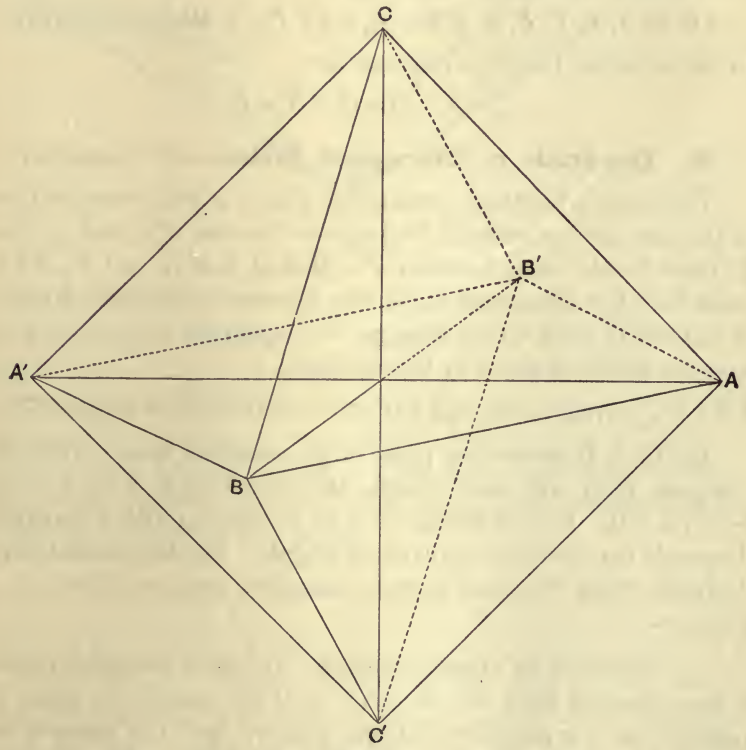

Fig. 6

planes of the form, and the complete crystal form is the octahedron with rectangular diagonals of different lengths.

The types of the possible hemihedral forms are:

1°. The tetrahedron formed by the planes ABC, $A'BC'$, $A'B'C$, $AB'C'$, where AA', BB' and CC' are the axes. This has no plane of symmetry but the figure will come into a similar position with respect to the axes after a rotation through two right angles about either axis, hence for these inclined-faced hemihedral forms formula (21) holds.

2°. The half-form whose planes are ABC, ABC', $A'B'C$, $A'B'C'$. This is identical with the complete monoclinic form, and may therefore be rejected.

3°. The half-form whose planes are ABC, $A'BC$, $A'B'C$, $AB'C$. This has two planes of symmetry, and formula (21) holds.

We shall write (21) in the form

$$2W = (A, B, C, F, G, H \mid e_0, f_0, g_0)^2 + La_0^2 + Mb_0^2 + Nc_0^2 \ldots (21).$$

In this notation Cauchy's relations are

$$L = F, \quad M = G, \quad N = H.$$

36. Quadratic or Tetragonal System—(6 Constants).

This system has three rectangular planes of symmetry, and two of the axes are equivalent; let these be the axes of x_0 and y_0, then P_0 must be the same function of e_0 that Q_0 is of f_0, and P_0 and Q_0 must have the same term in g_0; also S_0 must be the same function of a_0 that T_0 is of b_0; we thus get the equalities $a_{11} = a_{22}$, $a_{13} = a_{23}$, $a_{44} = a_{55}$, and W is given by the equation

$$2W = (a_{11}, a_{11}, a_{33}, a_{23}, a_{23}, a_{12} \mid e_0, f_0, g_0)^2 + a_{44}(a_0^2 + b_0^2) + a_{66}c_0^2 \ldots (22).$$

Let (h, k, l) denote any plane of the complete form. Then the complete form will also contain the planes $(\pm h, \pm k, \pm l)$ and $(\pm k, \pm h, \pm l)$. If $k = h$ the figure is an octahedron with rectangular diagonals two of which are equal in length. The hemihedral forms derivable from the most general complete form are obtained as follows:—

1°. Selection by alternate planes. Of the 8 bounding planes, 4 are obtained from the form (h, k, l) by taking the signs all positive, or two negative and one positive, and the other 4 are obtained from these by interchanging h and k and at the same time

changing the sign of l. The resulting inclined-faced hemihedral form has no plane of symmetry, but by rotating it round the z_0 axis through 90° it will come into a similar position with respect to the axes. We shall see presently that equation (22) does not hold for this case (fig. 7).

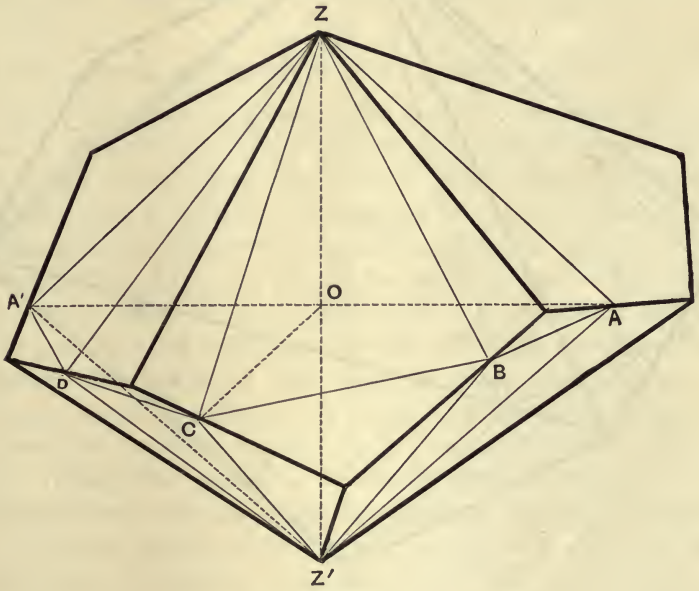

Fig. 7.

In fig. 7 AB and $A'D$ are two sides of a rhombus, two of whose corners are A, A', and the other two are on OC equidistant from O. BC and CD are two sides of an equal rhombus, one of whose corners is C, another on OC opposite to C, and the other two on AA'. The complete form consists of two pyramids, vertices Z and Z' standing on the 8 sided figure $ABCDA'\ldots$ The hemihedral forms obtained by the first method of selection are such as that bounded by the planes $Z'AB$, ZBC, $Z'CD$, $ZDA'\ldots$

2°. Selection by alternate pairs intersecting in the principal plane of symmetry (x_0 y_0). Of the 8 bounding planes, 4 are obtained from (h, k, l) by taking the first two letters either (h, k) or ($-h$, $-k$) or (k, $-h$) or ($-k$, h), and the other 4 from these by changing the sign of l. The resulting parallel-faced hemihedral form has three planes of symmetry, viz.: the plane (x_0, y_0) and

planes parallel to the sides of the square base in this plane, and formula (22) holds for these forms (fig. 8).

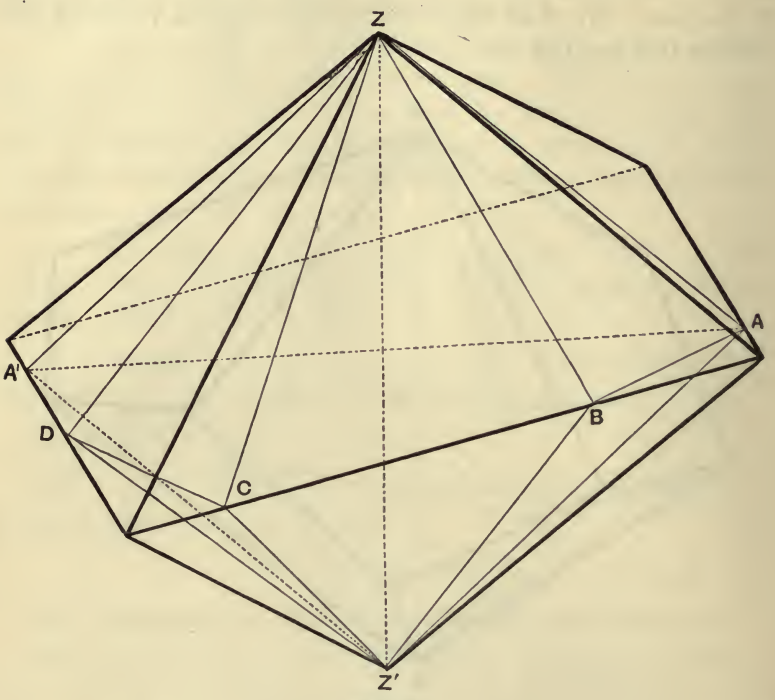

Fig. 8.

3°. Selection by alternate octants. Of the 8 bounding planes, 4 are obtained by taking the signs of (h, k, l) all positive or two negative and one positive, and the other 4 are obtained from these by interchanging h and k. The resulting inclined-faced hemihedral forms have two planes of symmetry which bisect the angles between the planes (x_0, z_0) and (y_0, z_0), but the equivalence of the corresponding axes of symmetry is lost. The form therefore belongs really to the rhombic system and may be rejected here.

For the trapezohedral hemihedrons obtained by the first method of selection it can be shewn that the energy-function is given by an equation of the form

$$2W=(a_{11}, a_{11}, a_{33}, a_{23}, a_{23}, a_{12}\chi e_0, f_0, g_0)^2+a_{44}(a_0^2+b_0^2)+a_{66}c_0^2+2a_{16}(e_0-f_0)c_0 \quad \ldots\ldots\ldots\ldots(23),$$

which involves 7 constants. This mode of crystallization has been observed in certain organic salts only.

37. Regular or Cubic System—(3 Constants).

This system has three rectangular planes of symmetry and all three axes equivalent. It is clear that W is given by the equation

$$2W = (a_{11}, a_{11}, a_{11}, a_{12}, a_{12}, a_{12} \chi e_0, f_0, g_0)^2 + a_{44}(a_0^2 + b_0^2 + c_0^2) \ldots (24).$$

Let (h, k, l) denote any plane of the crystal. Then the complete form is obtained by taking the six permutations of the letters h, k, l, and giving either sign to each letter. The most general complete form is therefore bounded by 48 planes. The types of hemihedrism are similar to those of the tetragonal system. If the first method (by alternate planes) be adopted, the resulting figure will have no plane of symmetry; but it will coincide with its original position after a rotation through 90° about either axis, and equation (24) holds for this case. If the second method (by alternate pairs intersecting in a principal plane of symmetry) be adopted, the resulting parallel-faced hemihedral forms will have three rectangular planes of symmetry, one belonging to the complete form, and the other two bisecting the angles between two principal planes of the complete form, and all three axes equivalent, and equation (24) will clearly hold for this case. If the third method (by alternate octants) be adopted the resulting inclined-faced hemihedral forms will be such that, by a rotation through 45° about either axis, the two principal planes of the complete form, that meet in that axis, become planes of symmetry, and equation (24) will therefore hold for this case.

Fluor-spar, Rock-salt, Pyrites, and Potassium Chloride are examples of minerals for which formula (24) holds.

38. Hexagonal System—(5 Constants).

This system has 7 planes of symmetry, of which one is perpendicular to the axis z_0, and 6 meet in the axis z_0 and are symmetrically arranged round that axis, and the axes perpendicular to the latter 6 planes are equivalent. We can express this by beginning with the rhombic system, and supposing that the expressions for the stresses in terms of the strains are unaltered by a rotation

through 60° about the axis z_0. When the new axes are given by the equations

$$\left.\begin{array}{lll} l_1=\frac{1}{2}, & m_1=\frac{1}{2}\sqrt{3}, & n_1=0 \\ l_2=-\frac{1}{2}\sqrt{3}, & m_2=\frac{1}{2}, & n_2=0 \\ l_3=0, & m_3=0, & n_3=1 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (25),$$

the equations of transformation of strain-components, given in (33) of art. 10, become

$$\left.\begin{array}{ll} e'=\frac{1}{4}e+\frac{3}{4}f+\frac{1}{4}\sqrt{3}c, & a'=\frac{1}{2}a-\frac{1}{2}\sqrt{3}b, \\ f'=\frac{3}{4}e+\frac{1}{4}f-\frac{1}{4}\sqrt{3}c, & b'=\frac{1}{2}\sqrt{3}a+\frac{1}{2}b, \\ g'=g, & c'=-\frac{1}{2}\sqrt{3}e+\frac{1}{2}\sqrt{3}f-\frac{1}{2}c \end{array}\right\} \ldots (26).$$

The equations of transformation of stress-components given in (14) of art. 16, give $P', Q', \ldots$ in terms of $P, Q, \ldots$ If we write down the corresponding formulæ for $P, Q, \ldots$ in terms of $P', Q', \ldots$ we shall get

$$\left.\begin{array}{ll} P=\frac{1}{4}P'+\frac{3}{4}Q'-\frac{1}{2}\sqrt{3}\,U', & S=\frac{1}{2}S'+\frac{1}{2}\sqrt{3}T', \\ Q=\frac{3}{4}P'+\frac{1}{4}Q'+\frac{1}{2}\sqrt{3}U', & T=-\frac{1}{2}\sqrt{3}\,S'+\frac{1}{2}T', \\ R=R', & U=\frac{1}{4}\sqrt{3}P'-\frac{1}{4}\sqrt{3}Q'-\frac{1}{2}U' \end{array}\right\} \ldots (27).$$

Now writing equation (21) in the form

$$2W=(A, B, C, F, G, H)(e_0, f_0, g_0)^2+La_0^2+Mb_0^2+Nc_0^2,$$

substituting for S' and T' in the S, T equations of (27), and equating coefficients of a or b, we obtain $L=M$.

Substituting for R' in the R equation of (27), and equating coefficients of e or f, we obtain $F=G$.

Substituting for P', Q', U' in the P, Q equations of (27), and equating coefficients of e in the P equation, and coefficients of f in the Q equation, we get $\frac{3}{4}N=\frac{15}{16}A-\frac{9}{16}B-\frac{3}{8}H$, and $\frac{3}{4}N=\frac{15}{16}B-\frac{9}{16}A-\frac{3}{8}H$, from which $A=B$, and $N=\frac{1}{2}(A-H)$.

Thus the energy-function is given by

$$2W=(a_{11}, a_{11}, a_{33}, a_{23}, a_{23}, a_{12})(e_0, f_0, g_0)^2+a_{44}(a_0^2+b_0^2)+\tfrac{1}{2}(a_{11}-a_{12})c_0^2 \quad \ldots\ldots\ldots\ldots (28).$$

Note that this formula is unaltered by turning the axes of x_0 and y_0 through any angle.

Beryl is an example of a crystal for which this formula holds.

39. **Rhombohedral System**—(6 Constants).

The most important hemihedral forms of the hexagonal system are the rhombohedrons obtained from a hexagonal pyramid by

the selection of alternate planes. In figure 9 $ABCA'B'C'$ is a regular hexagon, and ZZ' a perpendicular axis, and the faces of the rhombohedron are ZAB, $Z'BC$, ZCA', $Z'A'B'$, $ZB'C'$, and $Z'C'A$. These forms are unaltered by rotation through 120° about the axis z_0, and also by rotation through 180° about the line AA', which we take for the axis x_0. We have already seen (art. 34) that the last property produces just the same simplification in the energy-function as if the plane $x_0 = 0$ were a plane of symmetry, and we may therefore set out from the form

$$2W = (a_{11}, a_{22}, a_{33}, a_{44} \ldots \chi e_0, f_0, g_0, a_0)^2 + (a_{55}, a_{66}, a_{56} \chi b_0, c_0)^2.$$

If we work out the conditions that this may be unaltered by a

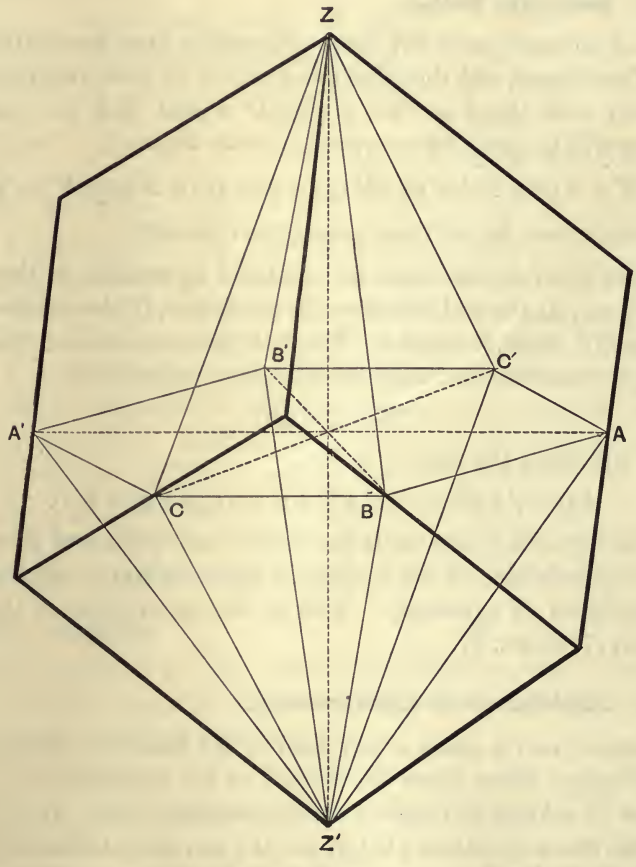

Fig. 9.

rotation through 120° about the axis z_0, we shall find the energy-function for this crystal system given by

$$2W = (a_{11}, a_{11}, a_{33}, a_{23}, a_{23}, a_{12} \mid e_0, f_0, g_0)^2 + a_{44}(a_0^2 + b_0^2) + \tfrac{1}{2}(a_{11} - a_{12})c_0^2 + 2a_{14}a_0(e_0 - f_0) + 2a_{14}b_0c_0 \quad \ldots\ldots\ldots\ldots (29).$$

After what we have just done for the holohedrons of the hexagonal system, the work may be left to the reader. Formula (29) holds for Iceland Spar.

Quartz is an example of a tetartohedral form of the hexagonal system, which is a hemihedral form of the rhombohedral system, and has the property of being unaltered by the same rotations as the rhombohedron. Formula (29) therefore holds for this mineral.

40. Isotropic Solids.

In an isotropic solid any three rectangular lines are equivalent axes of symmetry, and therefore there cannot be more independent constants than there are for a regular crystal, and the energy-function will be given by an equation of the form

$$2W = A(e^2 + f^2 + g^2) + 2H(fg + ge + ef) + N(a^2 + b^2 + c^2),$$

where there may be relations among the constants.

Again this function must be unaltered by rotation of the axes through any angle, and therefore, in particular, if the rotation be through 60° about the axis z. This will give us the same relation among the constants as holds for hexagonal crystals viz.:

$$H = A - 2N.$$

Hence $2W$ takes the form

$$A(e + f + g)^2 + N[a^2 + b^2 + c^2 - 4(fg + ge + ef)].$$

Now the quantities that occur herein are invariants, and therefore no further reduction in the number of constants can be effected by considerations of symmetry. This is the same form as that of equation (7) of art. 27.

41. Resistance to Compression.

Consider now a prism of any solid in the form of a rectangular parallelepiped whose edges are parallel to the coordinate axes, and suppose it subject to uniform surface-tractions only. It is clear that the stress-equations (10) of art. 14 can be satisfied by supposing P, Q, R all constant, and S, T, U all zero, and then, by

(12) of the same article, it follows that the surface-tractions are P, Q, R on the three pairs of faces.

Now let $P = Q = R = -p$, then a uniform pressure p is applied to the faces of the solid, and we have in general

$$\left.\begin{aligned}
c_{11}e + c_{12}f + c_{13}g + c_{14}a + c_{15}b + c_{16}c &= -p \\
c_{12}e + c_{22}f + c_{23}g + c_{24}a + c_{25}b + c_{26}c &= -p \\
c_{13}e + c_{23}f + c_{33}g + c_{34}a + c_{35}b + c_{36}c &= -p \\
c_{14}e + c_{24}f + c_{34}g + c_{44}a + c_{45}b + c_{46}c &= 0 \\
c_{15}e + c_{25}f + c_{35}g + c_{45}a + c_{55}b + c_{56}c &= 0 \\
c_{16}e + c_{26}f + c_{36}g + c_{46}a + c_{56}b + c_{66}c &= 0
\end{aligned}\right\} \quad \ldots\ldots\ldots(30).$$

Let Π be the determinant formed with the c's, and C_{rs} the minor of c_{rs}, then

$$e = -p\,(C_{11} + C_{12} + C_{13})/\Pi,$$
$$f = -p\,(C_{12} + C_{22} + C_{32})/\Pi,$$
$$g = -p\,(C_{13} + C_{23} + C_{33})/\Pi.$$

Hence $$p = -(e+f+g)\,\frac{\Pi}{C_{11} + C_{22} + C_{33} + 2C_{23} + 2C_{31} + 2C_{12}}.$$

If $$k \equiv \Pi/(C_{11} + C_{22} + C_{33} + 2C_{23} + 2C_{31} + 2C_{12}) \ldots\ldots\ldots\ldots(31),$$

then k is the ratio of the uniform pressure applied to the cubical compression produced. This is the resistance to compression, or *bulk-modulus* of the solid, for the set of directions (x, y, z). Now in the case under consideration the stress-quadric is a sphere $p(x^2 + y^2 + z^2) = \text{const.}$, and therefore if we transform to new axes the normal stresses will each be $-p$, and the tangential stresses will be each zero. Also we know that $e + f + g$ is an invariant. It follows that the bulk-modulus k is independent of the set of directions (x, y, z). It can be shewn without difficulty that, if uniform pressure p be applied at all points of the surface of any solid, uniform cubical compression p/k will be produced.

42. Rigidity.

Suppose that all the stress-components are zero except S; the stress reduces to a simple shearing stress of the planes (y, z), and if a be the shear produced, then

$$a = SC_{44}/\Pi.$$

Thus Π/C_{44} is the resistance to shear of the planes $y = \text{const.}$ parallel to the axis z, or of the planes $z = \text{const.}$ parallel to the axis y, and this may be called the *rigidity* for the directions (y, z).

In the case of crystals of the rhombic system the three rigidities, for the three pairs of principal directions, are the L, M, N of (21), p. 83 and a like simplification has place in the case of any body having three rectangular planes of symmetry.

In general suppose the stress-system to reduce to shearing stress S' of planes (l_2, m_2, n_2) and (l_3, m_3, n_3) at right angles to each other (see art. 17). Let (l_1, m_1, n_1) be the intersection of these planes, then the stress-system referred to the (x, y, z) axes will be

$$\left.\begin{array}{lll} P = 2l_2l_3S', & Q = 2m_2m_3S', & R = 2n_2n_3S' \\ S = (m_2n_3 + m_3n_2)\,S', & T = (n_2l_3 + n_3l_2)\,S', & U = (l_2m_3 + l_3m_2)\,S' \end{array}\right\} \quad (32).$$

The shear a' of the lines (2, 3) is the cosine of the angle between them after strain, and this is

$$a' = 2\,(l_2l_3e + m_2m_3f + n_2n_3g) + (m_2n_3 + m_3n_2)\,a + (n_2l_3 + n_3l_2)\,b + (l_2m_3 + l_3m_2)\,c \ldots (33).$$

Also $e = (C_{11}P + C_{12}Q + C_{13}R + C_{14}S + C_{15}T + C_{16}U)/\Pi$,

and f, g, ... are given by similar equations. Hence

$$\Pi a'/S' = (C_{11}, C_{22} \ldots C_{12} \ldots \}2l_2l_3,\ 2m_2m_3,\ 2n_2n_3,\ m_2n_3 + m_3n_2,\ n_2l_3 + n_3l_2,\ l_2m_3 + l_3m_2)^2 \quad \ldots\ldots\ldots (34),$$

where the right-hand side is a complete quadratic function of six arguments, and its coefficients are the minors of the c's in the determinant Π. The quantity obtained by dividing Π by the right-hand side of (34) is the rigidity for the directions (l_2, m_2, n_2), (l_3, m_3, n_3).

For a solid with three rectangular planes of symmetry, the rigidity for directions (2, 3) is the reciprocal of the expression

$$\frac{4(BC - F^2,\ CA - G^2,\ AB - H^2,\ GH - AF,\ HF - BG,\ FG - CH\}l_2l_3,\ m_2m_3,\ n_2n_3)^2}{(ABC + 2FGH - AF^2 - BG^2 - CH^2)} + \frac{1}{L}(m_2n_3 + m_3n_2)^2 + \frac{1}{M}(n_2l_3 + n_3l_2)^2 + \frac{1}{N}(l_2m_3 + l_3m_2)^2 \ldots\ldots\ldots (35).$$

The general formula (34) shews that there is some obscurity about the definition of the rigidity in art. 26, as the rigidity of a crystal depends, not only on the plane to which the tangential stress is applied, but also on the direction of the tangential stress applied to the plane. Suppose any two lines at right angles to each other drawn in the solid, and let these be taken as axes of y' and z' in a

system of rectangular axes of (x', y', z'), and let the stress be such that, referred to these axes, the stress-quadric is $2S'y'z' = \text{const.}$; then S' is a tangential stress on the plane $z' = 0$ parallel to the axis y', and there is an equal tangential stress upon the plane $y' = 0$ parallel to the axis z'. If the plane $z' = 0$ bound the solid, and the traction S' be uniformly applied to this plane face, we may expect that the planes $z' = \text{const.}$ will be made to move parallel to themselves, and to the axis y'. In the case of a crystal however this shear is only part of the effect, and it is by comparison of this shear with the traction S' that the "rigidity" for the directions (y', z') is arrived at. In regard to the formula (34) it is to be noticed that the rigidity for two directions is a symmetrical function of the direction-cosines defining them, *i.e.* it depends equally on the direction of displacement and on that of the normal to the planes displaced. It is easy to shew that in the case of an isotropic solid the formula becomes independent of both directions.

43. Young's Modulus.

Suppose that all the stress-components are zero except P, then

$$\Pi e = C_{11} P,$$

and Π/C_{11} is the ratio of the tension to the extension for the direction x. This is called the *Young's modulus* for this direction, and we shall denote it by E_1.

Suppose next that the stress-system reduces to simple tension P', in direction (l_1, m_1, n_1), then, referred to the axes of (x, y, z), we have

$$\left.\begin{array}{lll} P = l_1^2 P', & Q = m_1^2 P', & R = n_1^2 P' \\ S = m_1 n_1 P', & T = n_1 l_1 P', & U = l_1 m_1 P' \end{array}\right\} \quad \ldots\ldots\ldots(36).$$

The extension e' in this direction is

$$e' = e l_1^2 + f m_1^2 + g n_1^2 + a m_1 n_1 + b n_1 l_1 + c l_1 m_1 \ldots\ldots\ldots(37).$$

Hence

$$\Pi e'/P' = (C_{11}, C_{22}, \ldots C_{12} \ldots)(l_1^2, m_1^2, n_1^2, m_1 n_1, n_1 l_1, l_1 m_1)^2 \ldots(38),$$

where the right-hand side is a complete quadratic function of six arguments, whose coefficients are the minors of the c's in the determinant Π. The quantity obtained by dividing Π by the right-hand side of (38) is the Young's modulus of the material for the direction (l_1, m_1, n_1).

If we construct the surface of the fourth order

$$(C_{11}, C_{22}, \ldots C_{12} \ldots)(x^2, y^2, z^2, yz, zx, xy)^2 = \text{const.} \ldots\ldots (39),$$

the radius vector of this surface in any direction is inversely proportional to the real positive fourth root of the Young's modulus of the material for that direction.

When the solid possesses three rectangular planes of symmetry, the equation (38) may be written

$$\frac{1}{E} = \frac{l_1^4}{E_1} + \frac{m_1^4}{E_2} + \frac{n_1^4}{E_3} + \frac{2m_1^2 n_1^2}{F_1} + \frac{2n_1^2 l_1^2}{F_2} + \frac{2l_1^2 m_1^2}{F_3} \ldots\ldots (40),$$

where E_1, E_2, E_3 are the Young's moduluses for the three principal directions, and the E's and F's are given by such equations as

$$\frac{1}{E_1} = \frac{BC - F^2}{\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}}, \qquad \frac{2}{F_1} = \frac{2(GH - AF)}{\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}} + \frac{1}{L} \ldots (41).$$

This case has been discussed by Saint-Venant[1]. He shewed that there are in general 13 directions for which E becomes a maximum or minimum. Of these 3 are the axes of (x, y, z), 2 others lie in each of the coordinate planes between the axes, and the remaining 4 lie one in each of the trihedral angles formed by the coordinate planes. He also found that all these directions except the first three will be imaginary if

F_1 lie between E_2 and E_3,

F_2 lie between E_3 and E_1,

and F_3 lie between E_1 and E_2,

and if the 3 quantities such as

$$\left(\frac{1}{E_2} - \frac{1}{F_3}\right)\left(\frac{1}{E_3} - \frac{1}{F_2}\right) + \left(\frac{1}{F_3} - \frac{1}{F_1}\right)\left(\frac{1}{F_1} - \frac{1}{F_2}\right)$$

have not all the same sign.

In the notation of this art. the result (35) is that the rigidity for directions (2, 3) is the reciprocal of

$$4\left[\frac{l_2^2 l_3^2}{E_1} + \frac{m_2^2 m_3^2}{E_2} + \frac{n_2^2 n_3^2}{E_3} + \left(\frac{2}{F_1} - \frac{1}{L}\right) m_2 m_3 n_2 n_3 + \left(\frac{2}{F_2} - \frac{1}{M}\right) n_2 n_3 l_2 l_3 + \left(\frac{2}{F_3} - \frac{1}{N}\right) l_2 l_3 m_2 m_3\right]$$
$$+ \frac{(m_2 n_3 + m_3 n_2)^2}{L} + \frac{(n_2 l_3 + n_3 l_2)^2}{M} + \frac{(l_2 m_3 + l_3 m_2)^2}{N} \ldots\ldots (35),$$

[1] See the 'Annotated Clebsch,' pp. 95 sq.

and the result (31) of the article preceding is that the resistance to compression is the reciprocal of

$$\frac{1}{E_1}+\frac{1}{E_2}+\frac{1}{E_3}+\frac{2}{F_1}+\frac{2}{F_2}+\frac{2}{F_3}-\frac{1}{L}-\frac{1}{M}-\frac{1}{N} \quad \ldots\ldots\ldots(31).$$

44. Ratio of Lateral Contraction to Longitudinal extension.

When all the stresses except P vanish we have as before

$$\left.\begin{aligned} \Pi e &= C_{11}P \\ \Pi f &= C_{12}P \\ \Pi g &= C_{13}P \end{aligned}\right\} \quad \ldots\ldots\ldots(42).$$

Denote the ratio $-f/e$ in this case by σ_{12}, and $-g/e$ by σ_{13}, then $\sigma_{12} = -C_{12}/C_{11}$.

There are six principal quantities such as this, and they are the ratios of lateral contraction parallel to one axis to longitudinal extension parallel to another. We shall call them the *principal Poisson's ratios.*

In the case of three planes of symmetry, we have, with the notation of the last article,

$$\left.\begin{aligned} \sigma_{12} &= E_1\left(\frac{1}{2N}-\frac{1}{F_3}\right) \\ \sigma_{13} &= E_1\left(\frac{1}{2M}-\frac{1}{F_2}\right) \\ \sigma_{21} &= E_2\left(\frac{1}{2N}-\frac{1}{F_3}\right) \\ \sigma_{23} &= E_2\left(\frac{1}{2L}-\frac{1}{F_1}\right) \\ \sigma_{31} &= E_3\left(\frac{1}{2M}-\frac{1}{F_2}\right) \\ \sigma_{32} &= E_3\left(\frac{1}{2L}-\frac{1}{F_1}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots(43).$$

Observe that for a regular crystal

$$E_1 = E_2 = E_3 = E,\ F_1 = F_2 = F_3, \quad \text{and} \quad L = M = N.$$

Also for an isotropic solid the E's are equal to the F's, and the expressions for the bulk-modulus and the σ's and E's coincide with those found for k, σ, and E in arts. 27, 28.

45. Elastic Constants of Crystals.

The elastic constants of several minerals have been determined by Prof. Voigt of Göttingen by experiments on the torsion and flexure of rods. He shewed how to extend Saint-Venant's theories of torsion and flexure so as to apply to cases in which the æolotropy of the material was of a more general character than that of a rhombic crystal, having an axis of symmetry in the direction of the bar, but most of his experimental results depend upon the application of the theories of Saint-Venant. We give a résumé of some of Prof. Voigt's principal results, the constants being expressed in terms of a unit stress of 10^6 grammes' weight per square centimetre.

For Pyrites (regular), the constants are

$$a_{11} = 3680, \quad a_{44} = 1075, \quad a_{12} = -483,$$

and we have

Principal Young's modulus, $E = 3530$,

Principal Rigidity, $a_{44} = \mu = 1075$;

also by calculation we find Principal Poisson's ratio $\sigma = -\frac{1}{7}$ nearly.

These results are very remarkable, shewing that these moduluses of pyrites are much greater than those of steel[1], and further that a bar of the material cut in the direction of a principal axis when extended *expands* slightly in a lateral direction. The modulus of compression is about 1070×10^6 grammes' weight per square centimetre, which is considerably smaller than that of steel.

The table shews the values[2] of the constants for three other minerals for which the energy-function has the same form as for Pyrites. In this table a_{44} is the principal rigidity, and E is the principal Young's modulus.

Material	E	a_{11}	a_{12}	a_{44}
Fluor Spar	1470	1670	457	345
Rock-salt	418	477	132	129
Potassium Chloride	372	375	198	65·5

[1] See table, p. 77. The moduluses for different specimens of steel vary considerably.

[2] These numbers shew that there really exist materials which possess what Sir W. Thomson calls "cyboïd æolotropy." (See his *Lectures on Molecular Dynamics*, p. 158, and Note B at the end of this volume.)

Except in the case of Rock-salt, Cauchy's condition ($a_{12} = a_{44}$) is not even approximately verified, and the difference is much greater than could be accounted for by assuming experimental errors.

Beryl is a hexagonal crystal for which the constants are (art. 38),

$$a_{11} = 2746,\ a_{33} = 2409,\ a_{12} = 980,\ a_{23} = 674,\ a_{44} = 666.$$

For a bar whose axis is in the direction of the principal axis of symmetry $E = 2100$. For a bar whose axis is in the direction of a secondary axis of symmetry $E = 2300$. The first of these is the same as that for steel, and the second is rather greater. The principal rigidities are 666 and 980, of which the first is less and the second considerably greater than the rigidity of steel. Cauchy's relations are approximately verified.

Quartz is a trapezohedral tetartohedral form whose elastic character is similar to that of a rhombohedral crystal. The constants are

$$a_{11} = 868,\ a_{33} = 1074,\ a_{23} = 143,\ a_{12} = 70,\ a_{44} = 582,\ a_{14} = 171,$$

and E in the direction of the axis z_0 is 1030.

Topaz is a rhombic crystal whose principal Young's moduli and rigidities are greater than those of ordinary steel. The constants of formula (21) are for this mineral

$$A = 2870,\ B = 3560,\ C = 3000,\ F = 900,\ G = 860,\ H = 1280,$$
$$L = 1100,\ M = 1350,\ N = 1330,$$

the axis of x_0 being the brachydiagonal, and that of y_0 the macrodiagonal. The principal Young's moduli are 2300, 2890, 2650.

Barytes is a mineral whose energy-function has the same form, and its constants are

$$A = 907,\ B = 800,\ C = 1074,\ F = 273,\ G = 275,\ H = 468,$$
$$L = 122,\ M = 293,\ N = 283.$$

These results also shew that for these materials Cauchy's reduction is not valid.

46. Amorphous bodies.

There are many solids such as laminated metals, and fibrous woods, which exhibit a certain symmetry without being crystalline, *i.e.* there are three planes of symmetry, and the body may or may not be isotropic with reference to all lines drawn in one of them. In either case we may expect that the Young's modulus E will have only three (or two) maximum and minimum values,

and further we may expect that the Young's modulus in one principal direction will have a ratio neither very large nor very small to the principal rigidity for the pair of perpendicular axes, lastly, we may expect one at least of the ratios σ not to be very large. These conditions can of course be satisfied in a variety of ways. Saint-Venant proposes formulæ equivalent to

$$\left.\begin{aligned} P &= \frac{3MN}{L}e + Nf + Mg, & S &= La \\ Q &= Ne + \frac{3NL}{M}f + Lg, & T &= Mb \\ R &= Me + Lf + \frac{3}{\mathrm{n}}\frac{LM}{N}g, & U &= Nc \end{aligned}\right\} \quad \ldots\ldots\ldots(44),$$

where n is a number. These satisfy all the above conditions, and are put forward by Saint-Venant as likely to prove more useful in practical applications than the formulæ for an isotropic solid. (See the "Annotated Clebsch," p. 107.) In case the axes of x and y are interchangeable, we can simplify these formulæ by putting $L = M$.

It is easy to shew that

$$\frac{1}{\mathrm{n}} = 1 + \frac{8}{9}\left(\frac{E_3}{E_1}\frac{N^2}{L^2} - 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(45),$$

the work is left to the reader.

47. Determination of the constant n.

Saint-Venant has proposed an empirical formula for n in the form

$$\frac{1}{\mathrm{n}} = 1 + \frac{1}{\gamma}\left(\frac{E_3}{E_1} - 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(46),$$

where γ is a number. In the absence of definite evidence he proposed to put $\gamma = 16$. The ratio E_3/E_1 is known for some kinds of wood, *e.g.* it has been shewn by Hagen[1] that this ratio is 15 for oak, 22·5 for beech, 48 for pine, 83 for fir, the direction (3) being that of the fibres. If we eliminate n between equations (45) and (46), we obtain a formula for γ, viz.:

$$\frac{1}{\gamma} = \frac{8}{9}\left(\frac{E_3}{E_1}\frac{N^2}{L^2} - 1\right) \Big/ \left(\frac{E_3}{E_1} - 1\right) \quad \ldots\ldots\ldots\ldots (47).$$

Supposing the ratio E_3/E_1 known, this would enable us to find the value of γ by experiments on torsion directed to the deter-

[1] 'Sur l'élasticité du bois.' *Annales de Chimie*, XI. 1844.

mination of N and L. The same experiments would give n directly by (45).

48. Other distributions of Elasticity (Curvilinear Distributions).

There is another way in which the elasticity of a homogeneous solid may vary, viz.: two similar elements being cut from different portions of the solid, it may be necessary to have them differently orientated in order that they may exhibit similarity of elastic quality, the relative orientation depending on the relative position of the parts of the solid from which the elements are cut. An example will make this clearer. Suppose a thin plate of some material which has three planes of symmetry, one of which coincides with the plane of the plate, rolled into a circular cylinder; it will retain differences of quality depending on direction. A series of such plates may be put together so as to form a solid cylinder, and this may be done in such a way that the generators are axes of symmetry. Then, if similar elements be cut from two parts of the solid cylinder, they will exhibit similarity of elastic quality when turned so that the radii and generators of the cylindrical surfaces are parallel. This is an example of a "cylindrical distribution" of elasticity, and in like manner we might have spherical and other distributions. If we refer the matter in the neighbourhood of any point to a system of axes passing through the point, and chosen so as to simplify the stress-strain relations, the directions of the axes chosen will vary from point to point, and it will be best to use some system of curvilinear coordinates. We shall return to this matter when dealing with such systems. It is clear that processes of manufacture may introduce such variations of elasticity as we have just described, and we shall refer to them as "curvilinear distributions".

49. Note on Double Suffix Notations.

The kind of symmetry possessed by the sets of quantities that we have to consider is best brought out by the use of some form of double suffix notation. In the writings of different elasticians a great variety of different notations may be found. Some of them are tabulated and compared in art. 610 of Todhunter and Pearson's *History*, vol. I. All the notations that are really suggestive are clumsy. We have adopted Sir W. Thomson's notation, which is neat but not suggestive. It will be found

convenient to compare the notation employed with a double-suffix notation or some simplified form of one, and to keep the latter in mind.

The six *strains*

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial w}{\partial z}, \quad \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

may be denoted by $x_x,\ y_y,\ z_z,\ y_z,\ z_x,\ x_y$.

This is Kirchhoff's notation, and we have

$$e = x_x,\ f = y_y \ldots\ldots c = x_y.$$

The six *stresses* P, Q, R, S, T, U may be denoted by

$$\widehat{xx},\ \widehat{yy},\ \widehat{zz},\ \widehat{yz},\ \widehat{zx},\ \widehat{xy}.$$

Then *e.g.* $\widehat{xx}$ is the stress in direction x across a plane perpendicular to x, $\widehat{yz}$ is the stress in direction y across a plane perpendicular to z, and by Cauchy's theorem $\widehat{yz} = \widehat{zy}$. This is Prof. Pearson's notation.

The *elastic constants* may be expressed in the following manner. Write

$$\widehat{xx} = |xxxx|\, x_x + |xxyy|\, y_y + |xxzz|\, z_z + |xxyz|\, y_z + |xxzx|\, z_x + |xxxy|\, x_y,$$

$$\widehat{yz} = |yzxx|\, x_x + |yzyy|\, y_y + |yzzz|\, z_z + |yzyz|\, y_z + |yzzx|\, z_x + |yzxy|\, x_y,$$

and similarly for the other stresses, where the first two letters are always the same as those in the stress on the left, and the second two are always the same as those in the strain on the right. Now, from the nature of the quantities involved, it is clear that the first two letters in any of these coefficients may be interchanged, and in like manner the second two letters may be interchanged. With this understanding the number of the coefficients is 36. Green's reduction to 21 is effected by supposing that the first two letters may be interchanged with the second two; this is the relation $c_{rs} = c_{sr}$ of art. 31. Cauchy's further reduction to 15 will be arrived at by supposing that the order of the letters in any coefficient is indifferent.

To see the relation of the constants $|xxxx|$ .....with c_{11}.....we observe that the suffixes (1, 2, 3) correspond to the combinations of two letters which are alike (xx, yy, zz), and (4, 5, 6) to the combinations of two letters which are unlike (yz, zx, xy), thus we may write down the coefficient $c_{12} = |xxyy|$, and again $c_{66} = |xyxy|$. According to the rari-constant theory these two are identical, while the multi-constant theory makes them different (see Introduction).

CHAPTER IV.

THE STRENGTH OF MATERIALS.

50. For practical purposes it is important to have information as to the behaviour of materials strained beyond their elastic limits, and there exists a large body of knowledge of this kind which can be found in treatises on Applied Mechanics[1]. We propose in this chapter to give a brief account of some of the principal facts experimentally ascertained, and to explain in what sense the mathematical Theory of Elasticity becomes of use in applications to practice.

51. Stress-strain diagrams[2].

One of the greatest aids to scientific investigation of the properties of matter subjected to stress is the use of these diagrams. They are usually constructed by taking the strain developed as abscissa, and the stress producing it as the corresponding ordinate. For most materials the case selected for this kind of treatment is the extension of bars, and, in the diagram, the ordinate represents the traction applied, and the abscissa the elongation of the bar. The testing machine, by means of which the experiments are made, can be associated with an automatic recording apparatus[3] by which the curve is drawn. The general character of the curve for hard metals under extension, is now well known. It is for a considerable range of stress very nearly straight.

[1] *e.g.* Rankine's *Applied Mechanics*, Cotterill's *Applied Mechanics*, Unwin's *Testing of Materials of Construction.*

[2] The use of these diagrams appears to have been suggested by J. Bernoulli in 1694, but they were really introduced into practical treatises by Poncelet in 1839.

[3] See Prof. Unwin's treatise on the *Testing of Materials of Construction.*

Then comes a stage in which the curve is generally concave downwards, so that the strain increases faster than it would do if it were proportional to the traction; in this stage the strain is largely a permanent set. As the traction increases there comes a region of well-marked discontinuity, in which a small increase of traction produces a large increase of set. The traction at the beginning of this region is called the *Yield-Point*[1]. If the traction be increased above the yield-point, local flow of the material generally takes place at some point of the bar, determined apparently by accidental circumstances, and the bar ultimately breaks.

52. Elastic Limits.

The strain is in general wholly elastic, provided it be not greater than a certain limit, *i.e.* if a certain limiting stress be not exceeded, the strained body recovers its previous form and dimensions on the removal of the load. The elastic limits are by no means constant, even for the same kind of material, but appear to depend very much upon the previous history of the body tested. Cast iron is an example of a material which, in the natural state, cannot be subjected to any finite stress without receiving a permanent set. It is well known that the elastic limits may be gradually raised by the application of loads that produce a permanent set. Thus a bar of cast iron, after having received a set by the application of a small finite load, may be strained by the application of an equal or slightly greater load, without taking any increase of set. It is possible that in cases of this kind the initial set consists in the removal of a state of initial strain, or, in other words, that the application of a load to the body is required to reduce it to a state of ease. It is important to remember that there are in general two elastic limits for the same kind of strain; thus, for most materials, the limiting elastic extension differs from the limiting elastic compression. It is also important to remember that the elastic limits may be considerably widened by the gradual application and removal of loads that produce small permanent sets. The experimental determination of elastic limits is beset by difficulties, among which not the least important is that the elastic

[1] The phenomenon appears to have been first noticed by Dufour in Geneva in 1824, in some observations on a bridge of iron wire. Poncelet was acquainted with it. The name 'Yield Point' is due to Prof. Pearson (Todhunter and Pearson, vol. I.).

limit is the greatest stress that produces no set, while experiment can only determine whether there is a set large enough to be measured, and therefore the determination depends upon the delicacy of the measuring apparatus employed.

53. Time-effects. Plasticity.

The length of time that a body has been subjected to considerable stress generally affects the strain produced, and the length of time that a strained body has been free from stress generally affects the extent of the elastic recovery. The latter effect was discovered by Weber[1] in 1835 and called by him *Elastische Nachwirkung* or *elastic after-working*, the former appears to have been first noted by Vicat[2] in 1834. When a body has been strained by a load surpassing the elastic limit, and is set free, part of the strain disappears at once, and the strain that does not at once disappear gradually diminishes. The body never returns to its primitive condition, and the ultimate deformation is the permanent set, the part of the strain that gradually disappears is called elastic after-strain. To produce the effect noted by Vicat very considerable stress is generally required. He found that wires held stretched, with a tension equal to one quarter of the breaking stress, retained the length to which this tension brought them throughout the whole time of his experiments (33 months), while similar wires stretched with a tension equal to half the breaking stress exhibited a notable gradual increase of extension. The gradual flow of solids under great stress, indicated by these experiments, has been made the subject of exhaustive investigation by M. Henri Tresca[3]. He found, in his experiments on the punching and crushing of metals, results which point to the conclusion that all solids when subjected to very great pressure ultimately flow, *i.e.* take a set which increases with the time. An account of some of these experiments is given by Prof. Unwin[4]. This capacity of solids to flow under great stress is called *plasticity*. A solid is said to be hard when the force required to produce con-

[1] *De fili Bombycini vi Elastica.* Göttingen, 1841. An off-print of a paper communicated to the *Königliche Gesellschaft der Wissenschaften zu Göttingen*, 1835, and practically translated in Poggendorff's *Annalen*, xxxiv. 1835 and liv. 1841.

[2] *Note sur l'allongement progressif du fil de fer soumis à diverses tensions. Annales des ponts et chaussées*, 1er *semestre*, 1834.

[3] *Mémoires...par divers savans.* Paris, Acad. xviii. 1868, and xx. 1872.

[4] *The Testing of Materials of Construction.*

siderable set is great, soft or plastic when it is small; for a fluid it may be as small as we please, provided it is continuously applied.

54. Viscosity of Solids.

"Viscosity" is a general term for all those properties of matter in virtue of which the resistance, which a body offers to any change, depends upon the rate at which the change is effected. The existence of viscous resistances involves a dissipation of the energy of the substance, the kinetic energy of molar motion being transformed, as is generally supposed, into kinetic energy of molecular agitation. The most marked effect of this property, in the case of elastic solids, is the dying away of vibrations set up in the solid. Suppose a solid of any form struck, or otherwise suddenly disturbed. It will be thrown into more or less rapid vibration, and the stresses developed in it will depend partly on the displacements, and partly on the rates at which they are effected. The parts of the stresses depending on these rates of change are viscous resistances, and they ultimately destroy the vibratory motion. This effect is quite different to the time-effects previously noticed. The latter are also influential in producing a decay of oscillations, and they also disclose a transformation of energy, but apparently it is the potential energy of strain that is transformed into some other form of energy, possibly as suggested by Maxwell[1] and Mr J. G. Butcher[2] into energy of separation of more or less stable groups of molecules. It is necessary to mention this distinction as Sir W. Thomson[3], unless read very carefully, appears to attribute the dying away of vibrations entirely to the *Elastische Nachwirkung* of Weber, while in Maxwell's *Heat* the term "viscosity" is used in a different sense, namely, that a body is more or less viscous, according as a change produced by the application of stress takes a greater or less time to produce.

55. Æolotropy induced by permanent set.

One of the changes produced in a solid, which has received a permanent set, may be that the material, previously isotropic, becomes æolotropic. The best known example is that of a bar rendered æolotropic by permanent torsion. Herr Warburg[4] found

[1] *Constitution of Bodies*, *Encycl. Brit.*

[2] *Proc. Lond. Math. Soc.* VIII. p. 103.

[3] *Elasticity*, *Encycl. Brit.* and *Math. and Phys. Papers*, vol. III. p. 27.

[4] Wiedemann's *Annalen*, x. 1880.

that, in a copper wire to which a permanent twist had been given, the elastic phenomena observed could all be explained on the supposition that the substance of the wire was rendered æolotropic like a rhombic crystal. The strain of the substance observed as a permanent set may be regarded as a shear of the material, the plane of the shear being the plane parallel to the axis and perpendicular to the axial radius vector at any point, and this shear is by art. 7 capable of analysis into a compression and an extension along the principal axes of the shear, which are the lines bisecting the angles between the axis of the wire and the direction of displacement of any point. The axes of the rhombic crystal whose æolotropy is similar to that of the substance of the wire are the axial radius vector and the principal axes of the shear, and it was found that the Young's modulus in the direction of the permanent extension was increased, that in the direction of the permanent compression diminished. When a weight was hung on the wire it produced, in addition to extension, a small shear, equivalent to a partial untwisting[1] of the wire; this was an elastic strain, and disappeared on the removal of the load. This experiment is important as shewing that processes of manufacture may induce considerable æolotropy in materials which in the unworked stage are isotropic, and consequently that estimates of strength, founded on the employment of the equations of isotropic elasticity, cannot be strictly interpreted.

56. Fatigue of Resistance.

A body strained within its elastic limits may be strained again and again without receiving any injury; thus a watch-spring may be coiled and uncoiled millions of times without deterioration. But it is different when a body is strained repeatedly by rapidly varying loads. Herr Wöhler's[2] experiments on this point have been held to shew that the resistance of a body to any kind of deformation can be seriously diminished, by rapidly repeated applications of a load. The result appears to point to a gradual deterioration[3] of the quality of the material subjected to repeated loading, which can be verified by the observation that after a

[1] Cf. Sir W. Thomson's article *Elasticity*, *Encycl. Brit.*, see *Math. and Phys. Papers*, vol. III. p. 82.

[2] *Ueber die Festigkeitsversuche mit Eisen und Stahl.* An account of Wöhler's methods and results is given in Unwin's *Testing of the Materials of Construction.*

[3] Prof. Pearson proposes a different interpretation. See *Mess. of Math.*, 1890.

large number of applications and removals of the load, bars may be broken by a stress, much below the statical breaking stress, and even below the elastic limit.

An analogous property of bodies is that to which Sir W. Thomson[1] has called attention under the name "fatigue of elasticity". He observed that the torsional vibrations of wires died away much more rapidly when the wires had been kept vibrating for several hours or days, than when, after being at rest for several days, they were set in vibration and immediately left to themselves.

Experimental results of this kind point to the importance of taking into account the manner and frequency of the application of force to a structure in estimating its strength.

57. Theories of Rupture.

Various theories have been advanced as to the conditions under which a body is ruptured, or a structure becomes unsafe. Thus Lamé[2] supposed it necessary that the greatest tension should be less than a certain limit. Poncelet[3], followed by Saint Venant[4], assumed that the greatest extension must be less than a certain limit. These measures of *tendency to rupture* agree for a bar under extension, but in general they lead to different limits of safe loading[5]. Again M. Tresca followed by Prof. G. H. Darwin[6] makes the maximum difference of the greatest and least principal stresses the measure of tendency to rupture, and a not very different limit would be found by following Coulomb's[7] suggestion, that the greatest shear produced in the material is a measure of this tendency. It is possible that the proper limit of safety depends on the nature of the load (whether it tend to produce extension or

[1] *Math. and Phys. Papers*, vol. III. p. 22.

[2] See *e.g.* the memoir of Lamé and Clapeyron, *Mém. par divers savans*, Paris Acad. vol. IV. 1833. The same supposition is made by a great many elasticians.

[3] See Todhunter and Pearson, vol. I. art. 995.

[4] See especially the *Historique Abrégé* in Saint-Venant's edition of the *Leçons de Navier*, pp. cxcix.—ccv.

[5] For examples see Todhunter and Pearson, vol. I. p. 550 footnote, and art. 150 of the present work.

[6] 'On the stresses produced in the interior of the Earth by the weight of Continents and Mountains', *Phil. Trans. R. S.* 1882. The same measure is adopted in the account of Prof. Darwin's work in Thomson and Tait's *Nat. Phil.* Part II. art. 832'.

[7] 'Essai sur une application des règles de Maximis &c.', *Mém. par divers Savans*, 1776. Introduction.

compression, or shear), and on the manner in which it is applied. The conditions of rupture are but vaguely understood, and may depend largely on these and other accidental circumstances. At the same time the question is very important, as a satisfactory answer to it might suggest in many cases causes of weakness previously unsuspected, and, in others, methods of economising material that would be consistent with safety. The answer given in most practical treatises is that the stress developed in the material must nowhere exceed a certain fraction of the breaking stress. The reciprocal of this fraction is called the "factor of safety", and it depends in general on the nature of the strain to which the structure or machine will be subjected. Thus a factor[1] 6 is allowed for boilers, 10 for pillars, 6 for axles, 6 to 10 for railway-bridges, and 12 for screw-propeller-shafts and parts of other machines subjected to sudden reversals of load.

In all these theories it is supposed that the stress or strain actually produced in a body of given form, by a given load, is somehow calculable. The only known method of calculating these effects is by the use of the mathematical theory of elasticity, or by some more or less rough and ready rule obtained from some result of this theory. To explain how this is done we may consider any one of the theories described above, *e.g.* that of Poncelet and Saint-Venant. According to this theory, rupture ensues, if at some point, and in some direction, the greatest principal extension exceed a certain limit, depending on the direction if the body be æolotropic. Suppose the body subject to a given system of load, and suppose that we know how to solve the equations of elastic equilibrium with the given boundary-conditions. The extension at every point in every direction can be found, and the maximum principal extension can be deduced; this maximum will be linear in the expression of the given load, and equating it to the limiting extension consistent with safety[2], we shall have an equation to determine the maximum safe load. Now the maximum referred to, being found by the use of the equations of elasticity, it is virtually

[1] My authorities for these numbers are Unwin's *Testing of Materials of Construction*, Rankine's *Applied Mechanics*, and Box's *Strength of Materials*.

[2] This limiting extension e is given by the equation $T_0 = \Phi E e$, where E is Young's modulus of the material for a given direction, T_0 is the breaking tension of a bar cut in that direction, and Φ is the factor of safety.

assumed that the strain is a small elastic strain, while, at the same time, it is perfectly well known that there is considerable set before rupture is approached. The only logical way out of the difficulty would be to use, instead of the elastic equations, others, in which set is properly taken into account, and these are unfortunately still unknown.

58. Practice and Theory.

The application of the Mathematical Theory of strains, so small that their squares are negligible, to practical problems of strength cannot, from the nature of the case, be exact. The theory is capable of determining exactly such small strains as those produced in the interior of the Earth assumed to be solid by the tidal disturbing attraction of the Sun and Moon, and the modes and periods of the free vibrations of solids. It is not capable of answering exactly any other kind of question than such as relate to very small strains. The results to which it leads, in other cases, are qualitative rather than quantitative. Thus, by Saint-Venant's theory of the torsion of prisms, it can be predicted that a shaft transmitting a couple by torsion is seriously weakened by the existence of a dent having a curvature approaching to that in a reentrant angle, or by the existence of a flaw parallel to the axis of the shaft. By the theory of equilibrium of a mass with a spherical boundary, it can be predicted that the shear in the neighbourhood of a flaw of spherical form may be as great as twice that at a distance. The result of such theories would be that a factor of safety must be allowed for shafts transmitting a couple in which such flaws may occur, and it suggests that 2 may be the factor required—this is the factor allowed in practice to guard against this kind of weakness; other factors are allowed for other reasons. Again, it can be shewn that, in certain cases, a load suddenly applied may cause a strain twice[1] as great as that produced by a gradual application of the same load, and that a load suddenly reversed may cause a strain three times as great as that produced by the gradual application of the same load. This shews that additional factors of safety will be required for sudden applications and sudden reversals, and it suggests that

[1] This point appears to have been first expressly noted by Poncelet in his *Introduction à la Mécanique industrielle, physique et expérimentale* of 1839, see Todhunter and Pearson, vol. I. art. 988.

these factors may be 2 and 3. The actual values that ought to be given to these factors in practice can however only be found by experiment. Again, a source of weakness in structures, some parts of which are very thin bars or plates subjected to thrust, is a possible buckling of the parts. The conditions of buckling can be determined from the theory of Elastic Stability, and this theory can thus be made to suggest some method of supporting the parts by stays, and the best places for them, so as to secure the greatest strength with the least expenditure of materials; but the result, at any rate in structures that may receive small permanent sets, is only a suggestion and requires to be verified by experiment.

There exists no adequate mathematical theory of set, or of after-strain, or in fact of any of the phenomena exhibited by materials strained beyond their elastic limits. No one apparently has ever attempted to formulate general equations of equilibrium or motion of a solid so strained, although empirical laws of set have been suggested, such as the Coulomb-Gerstner law[1]; and the researches[2] of Boltzmann, Kohlrausch, Meyer, Neesen and others have gone some way towards the foundation of a theory of after-strain. Yet it is imperatively necessary that effects which cannot be calculated exactly should be taken into account in constructions, and it is in this sense that elastic theory is at this time behind engineering practice.

[1] See *Handbuch der Mechanik*, by Franz Joseph Ritter von Gerstner. Prag, 1833; and Todhunter and Pearson, vol. I. p. 441.

[2] For a good account of the theory see F. Braun in *Encyclopädie der Naturwissenschaften, Handbuch der Physik*, Bd. I. pp. 321—342.

CHAPTER V.

GENERAL THEOREMS.

STRESS-STRAIN EQUATIONS DEDUCED FROM CAUCHY'S MOLECULAR HYPOTHESIS.

59. Statement of the Hypothesis.

We proceed to investigate, after the manner of Cauchy[1], the stress across a small plane area arising from the forces (supposed insensible at sensible distances), that act between the individual pairs of a system of particles homogeneously arranged, when the force between two whose masses are m and m' placed at a distance r is $mm'\chi(r)$. The nature of the homogeneity of the arrangement can be described by stating that all the particles have the same mass, and, if P, P', Q be the positions of any three particles, and a line QQ' be drawn from Q equal and parallel to PP', there will be a particle at Q'.

60. Evaluation of stress-components.

Now, as in ch. I., let x, y, z be the coordinates before strain of a point P, at which there is a particle of mass M, and $x+\xi$, $y+\eta$, $z+\zeta$ those of another point Q at which there is a particle m of equal mass, r the distance between them, and λ, μ, ν the direction-cosines of PQ, so that

$$\xi = \lambda r, \quad \eta = \mu r, \quad \zeta = \nu r \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Through P draw a plane parallel to the plane yz, and let m_1, m_2,... denote particles on the side of the plane where x is

[1] 'De la pression ou tension dans un système de points matériels'. *Exercices de Mathématiques*, 1828.

greater than the x of P, and m_1', m_2',... particles on the other side of the plane. Describe round P any small curve in the plane $x = \text{const.}$ through P whose radii vectores are all sensibly greater than the greatest distance at which the force between two particles is sensible, and let s be the area of this curve; then, in the notation of ch. II., if

$$P.s,\ U.s,\ T.s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

be the sums of components, parallel to x, y, z, of all the forces that cross the plane within the curve s, P, U, T will be the component stresses at P across a plane parallel to x. Now these sums of components are the sums of such quantities as

$$m_i m_j' \chi(r_{ij}) \lambda_{ij},\quad m_i m_j' \chi(r_{ij}) \mu_{ij},\quad m_i m_j' \chi(r_{ij}) \nu_{ij} \ldots\ldots(3),$$

where m_i and m_j' are the masses of two particles on opposite sides of the plane, r_{ij} the distance between them, and λ_{ij}, μ_{ij}, ν_{ij}, the direction-cosines of this line, and the summation must be extended to all pairs so situated that the line joining them crosses s, and the distance between them does not exceed the greatest distance at which the force is sensible (called by Cauchy the "radius of the sphere of molecular activity").

Now there will be a particle m whose distance r from M is r_{ij}, and such that the line joining M, m is parallel to the line joining m_i, m_j', and therefore the force across s arising from the force between m_i and m_j' will have components

$$Mm\chi(r)\lambda,\quad Mm\chi(r)\mu,\quad Mm\chi(r)\nu \ldots\ldots\ldots\ldots(4).$$

The summation may be taken by first summing for all the pairs of particles (m_i, m_j') that have the same r, λ, μ, ν, and are so situated that the line joining them crosses s, and then summing for all the directions λ, μ, ν on which pairs of particles are met with, and lastly summing for all the particles on each such line (λ, μ, ν) whose distance apart is not greater than the radius of the sphere of molecular activity. The first summation will be made by multiplying the expressions (4) by the number of particles contained in a cylinder standing on s whose height is $r\lambda$; this number is $\rho s r\lambda/M$, where ρ is the density, or mass per unit volume, of the system of particles, and thus we get for the component stresses per unit area across the plane parallel to (yz) through M, the sums of such quantities as

$$\rho m r \lambda^2 \chi(r),\quad \rho m r \lambda\mu\chi(r),\quad \rho m r \lambda\nu\chi(r) \ldots\ldots\ldots\ldots(5).$$

Now it is clear that, if the summation be extended to all directions round M in which particles are met with, the force between any pair m_i, m_j' will have been counted twice, and we thus get

$$P=\tfrac{1}{2}\rho\Sigma[mr\lambda^2\chi(r)],\ U=\tfrac{1}{2}\rho\Sigma[mr\lambda\mu\chi(r)],\ T=\tfrac{1}{2}\rho\Sigma[mr\lambda\nu\chi(r)]\ldots(6),$$

where the summations refer to all particles m, whose distance from M is not greater than the radius of the sphere of molecular activity.

61. Stress in terms of strain.

Now let the system be displaced so that M comes to

$$(x+u,\ y+v,\ z+w),$$

and m comes to

$$(x+u+\xi+\delta u,\ \ y+v+\eta+\delta v,\ \ z+w+\zeta+\delta w),$$

then, since m is very near to M, we may express δu, δv, δw in the forms

$$\left.\begin{aligned}\delta u&=\xi\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z}\\ \delta v&=\xi\frac{\partial v}{\partial x}+\eta\frac{\partial v}{\partial y}+\zeta\frac{\partial v}{\partial z}\\ \delta w&=\xi\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+\zeta\frac{\partial w}{\partial z}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(7),$$

and use the notation e, f, g, a, b, c, Δ of strain-components.

Let r become $r(1+\epsilon)$, then, by (33) of art. 11,

$$\epsilon=e\lambda^2+f\mu^2+g\nu^2+a\mu\nu+b\nu\lambda+c\lambda\mu\ldots\ldots\ldots\ldots(8).$$

Also $r\lambda$ is the difference of the x's of m and M, and this becomes

$$\xi\left(1+\frac{\partial u}{\partial x}\right)+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z},$$

or

$$r\lambda+r\left(\lambda\frac{\partial u}{\partial x}+\mu\frac{\partial u}{\partial y}+\nu\frac{\partial u}{\partial z}\right)=r\lambda+\delta(r\lambda),\text{ say }\ldots\ldots(9),$$

and in like manner we may write down the values of $r\mu$, $r\nu$ after strain.

The new value of $\chi(r)$ is

$$\chi(r)+\epsilon r\chi'(r)\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

The new value of ρ is

$$\rho[1-(e+f+g)]=\rho'\text{ say }\ldots\ldots\ldots\ldots\ldots(11).$$

Thus P, U, T become

$$\left.\begin{aligned} P &= \tfrac{1}{2}\rho'\Sigma\left[m\frac{1}{r(1+\epsilon)}\{\chi(r)+\epsilon r\chi'(r)\}\{r\lambda+\delta(r\lambda)\}^2\right] \\ U &= \tfrac{1}{2}\rho'\Sigma\left[m\frac{1}{r(1+\epsilon)}\{\chi(r)+\epsilon r\chi'(r)\}\{r\lambda+\delta(r\lambda)\}\{r\mu+\delta(r\mu)\}\right] \\ T &= \tfrac{1}{2}\rho'\Sigma\left[m\frac{1}{r(1+\epsilon)}\{\chi(r)+\epsilon r\chi'(r)\}\{r\lambda+\delta(r\lambda)\}\{r\nu+\delta(r\nu)\}\right] \end{aligned}\right\} \quad (12).$$

We shall put down P and U, we get

$$\begin{aligned} P &= \tfrac{1}{2}\rho'\Sigma\,[mr\lambda^2\chi(r)] \\ &\quad + \rho'\left\{\Sigma\,[mr\lambda^2\chi(r)]\frac{\partial u}{\partial x} + \Sigma\,[mr\lambda\mu\chi(r)]\frac{\partial u}{\partial y} + \Sigma\,[mr\lambda\nu\chi(r)]\frac{\partial u}{\partial z}\right\} \\ &\quad + \tfrac{1}{2}\rho'\Sigma[mr\{r\chi'(r)-\chi(r)\}\lambda^2(e\lambda^2+f\mu^2+g\nu^2+a\mu\nu+b\nu\lambda+c\lambda\mu)] \quad \ldots(13). \end{aligned}$$

$$\begin{aligned} U &= \tfrac{1}{2}\rho'\Sigma\,[mr\lambda\mu\chi(r)] \\ &\quad + \tfrac{1}{2}\rho'\left\{\Sigma\,[mr\lambda^2\chi(r)]\frac{\partial v}{\partial x} + \Sigma\,[mr\lambda\mu\chi(r)]\frac{\partial v}{\partial y} + \Sigma\,[mr\lambda\nu\chi(r)]\frac{\partial v}{\partial z}\right\} \\ &\quad + \tfrac{1}{2}\rho'\left\{\Sigma\,[mr\mu\lambda\chi(r)]\frac{\partial u}{\partial x} + \Sigma\,[mr\mu^2\chi(r)]\frac{\partial u}{\partial y} + \Sigma\,[mr\mu\nu\chi(r)]\frac{\partial u}{\partial z}\right\} \\ &\quad + \tfrac{1}{2}\rho'\Sigma\,[mr\{r\chi'(r)-\chi(r)\}\lambda\mu(e\lambda^2+f\mu^2+g\nu^2+a\mu\nu+b\nu\lambda+c\lambda\mu)] \quad (14). \end{aligned}$$

In like manner the other four stresses can be put down.

Now suppose the initial state of the system is one of zero stress, or that the system is disturbed from the natural state, then we see that all the 6 quantities such as

$$\Sigma\,[mr\lambda^2\chi(r)], \quad \Sigma\,[mr\lambda\mu\chi(r)] \quad \ldots\ldots\ldots\ldots\ldots(15)$$

must vanish identically, and, therefore, the expressions of the six stresses in terms of the strains are such quantities as the last lines of the right hand sides of (13) and (14). In these, neglecting squares of the strains, we may put ρ for ρ', and thus writing for shortness

$$r\{r\chi'(r)-\chi(r)\} = \phi(r) \quad \ldots\ldots\ldots\ldots\ldots(16),$$

we find such expressions as

$$\left.\begin{aligned} P &= \tfrac{1}{2}\rho\Sigma\,[m\phi(r)\lambda^2\ (e\lambda^2+f\mu^2+g\nu^2+a\mu\nu+b\nu\lambda+c\lambda\mu)] \\ U &= \tfrac{1}{2}\rho\Sigma\,[m\phi(r)\lambda\mu\,(e\lambda^2+f\mu^2+g\nu^2+a\mu\nu+b\nu\lambda+c\lambda\mu)] \end{aligned}\right\} \quad \ldots(17).$$

Hooke's Law follows at once, and the elastic constants are such expressions as

$$\begin{aligned} &\tfrac{1}{2}\rho\Sigma\,[m\phi(r)\lambda^4], \quad \tfrac{1}{2}\rho\Sigma\,[m\phi(r)\lambda^3\mu], \\ &\tfrac{1}{2}\rho\Sigma\,[m\phi(r)\lambda^2\mu^2], \quad \tfrac{1}{2}\rho\Sigma\,[m\phi(r)\lambda^2\mu\nu] \quad \ldots\ldots\ldots(18), \end{aligned}$$

and there are 15 of these.

If all the stress-equations similar to the above be written down, and the coefficients compared with the elastic constants c of art. 31, it will be found that

$$c_{rs} = c_{sr}, \quad (r, s = 1, 2 \ldots 6),$$

and that

$$\left.\begin{aligned} c_{23} = c_{44}, \quad c_{31} = c_{55}, \quad c_{12} = c_{66} \\ c_{14} = c_{56}, \quad c_{25} = c_{64}, \quad c_{36} = c_{45} \end{aligned}\right\},$$

as in equations (20) of that article.

The particular result for isotropic solids is that $\lambda = \mu$, and consequently $\sigma = \frac{1}{4}$, as stated in art. 28.

62. The Thermo-Elastic Equations.

Consider a solid strained by unequal heating. Suppose that, when the temperature of any part is increased by t, the force between two particles m, m' is increased by a quantity of the form $mm'Kt$, where K is independent of the configuration. Then, referring to the investigation of art. 60, we see that we have to add to the expressions for the stresses the sum of all such quantities as

$$m_i m_j' Kt\lambda, \quad m_i m_j' Kt\mu, \quad m_i m_j' Kt\nu,$$

where m_i, m_j' are the equal masses of particles in a line crossing the area s; and, as before, the stresses thence arising are given by such equations as

$$P = \tfrac{1}{2}\rho\Sigma\,[mr\lambda^2 Kt], \quad U = \tfrac{1}{2}\rho\Sigma\,[mr\lambda\mu Kt].$$

We should find in this way the stresses given by such equations as (17), each increased by a quantity, which is the product of t and a constant depending on the material.

In case the particles of the system are distributed symmetrically in all directions, the terms contributed by t to the tangential stresses will disappear, and the terms contributed to the normal stresses will all be equal, so that the stresses will consist of

(1) a hydrostatic pressure proportional to the change of temperature,

(2) elastic stresses like those of (17) due to the strains.

The equations of equilibrium hence deduced will be three of the form

$$\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} + \rho X = \beta \frac{\partial t}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

where β is a constant, and $P \ldots$ are component stresses due to the strains, and are the same functions of the strains as occur when t is constant.

The boundary conditions will be three of the form

$$lP + mU + nT = F + l\beta t \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

These are the Thermo-elastic equations of Duhamel and Neumann, and the above method of obtaining them is practically that of Duhamel[1] and Weyrauch[2]. We may, however, obtain them by another method, dispensing with the hypothesis of material points with forces between them. To do this we shall have to make subsidiary assumptions.

1°. When a homogeneous isotropic solid is uniformly heated, and allowed to change form freely, a uniform cubical dilatation takes place proportional to the change of temperature.

2°. When the solid is heated and the volume of every part kept constant, stresses are developed in the interior, which, we shall assume, reduce to a hydrostatic pressure proportional to the change of temperature. This we shall call the thermal stress.

3°. When a solid is unequally heated the stress at any point consists of two parts:—(1) a hydrostatic pressure proportional to the change of temperature (the thermal stress), (2) elastic stresses proportional to the strains.

Of these assumptions 1° appears to be experimentally verified, when the changes of temperature do not amount to more than a few degrees Centigrade; 2° is in some sense a consequence of 1° since we know that in the case of elastic strains and stresses uniform pressure produces uniform cubical compression, and we have only to assume that this holds however the stress be produced; 3° is then the assumption that the stress due to strain and variation of temperature is the sum of the stress due to strain without variation of temperature, and the stress due to variation of temperature without strain.

This method leads to the same equations as Duhamel's method, it is practically that of F. E. Neumann[3]. The resulting equations cannot be regarded as proved, but only as resting on probable hypotheses. They are applied by Duhamel and Neumann to cases where t is a given function of x, y, z and the time.

[1] 'Mémoire sur le calcul des actions moléculaires...', *Mém....par divers Savans*, 1838.

[2] *Theorie der Elasticität fester Körper.*

[3] *Vorlesungen über die Theorie der Elasticität der festen Körper...*

63. Sir W. Thomson's Theorem of the Existence of the Energy-Function[1].

In the general case of an æolotropic solid, the stress-strain relations contain 36 constants, and we have seen that in the special case of isotropy these are reduced to 2, and *that* in such a way that the stresses are the partial differential coefficients of a quadratic function W of the strains, this function denoting the potential energy of strain. The method by which we obtained this result was founded on particular assumptions. If such a function exist in the general case, there will be 15 linear relations among the constants, reducing their number to 21. We shall now shew that, if the solid be strained either at constant temperature, or in such a way that no heat is allowed to escape from or to enter into any element, then the function W exists.

Let e, f, g, a, b, c be the six components of strain, and, in some state of the solid, let $e_0, f_0, g_0, a_0, b_0, c_0$ be the values of these quantities, and let t be the temperature of the solid measured on some absolute scale, *i.e.* on some scale independent of the working substance, and let the state of the solid defined by $e = e_0, \ldots a = a_0, \ldots t = t_0$ be taken as the standard state, then the intrinsic energy E of the solid is the whole work that would have to be done upon it to bring it from the state $e_0, f_0, g_0, a_0, b_0, c_0, t_0$ to the state e, f, g, a, b, c, t. Since the state of the solid depends only upon e, f, g, a, b, c, t it follows that E is some function of these quantities, say

$$E = \phi(e, f, g, a, b, c, t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

Then a knowledge of ϕ includes a knowledge of all the thermo-elastic properties of the solid.

Now let the solid be strained at constant temperature from the state $(e_0, f_0, g_0, a_0, b_0, c_0, t)$ to the state (e, f, g, a, b, c, t), and let H be the heat that must be supplied to it to keep its temperature constant, and then let the solid be brought back from the state (e, f, g, a, b, c, t) to the state $(e_0, f_0, g_0, a_0, b_0, c_0, t)$, through the same or any other series of states, provided it is always at

[1] *Quarterly Journal*, v. 1857.

constant temperature, and let H' be the heat supplied to it during this process. Then, by the second law of Thermo-dynamics,

$$\frac{H}{t}+\frac{H'}{t}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Hence $H=-H'$, and the heat that must be supplied to the solid to keep its temperature constant, while it is being strained by the action of external forces from the state $(e_0, f_0, g_0, a_0, b_0, c_0)$ to the state (e, f, g, a, b, c), is independent of the particular succession of states through which it is made to pass. Hence H is a function of (e, f, g, a, b, c) which vanishes when $e=e_0$.... so that we may take

$$H=\psi(e, f, g, a, b, c, t)-\psi(e_0, f_0, g_0, a_0, b_0, c_0, t)\ldots\ldots(23).$$

Let E_0 be the intrinsic energy in the state $(e_0, f_0, g_0, a_0, b_0, c_0, t)$, then the whole change of the intrinsic energy in passing from the state $(e_0, f_0, g_0, a_0, b_0, c_0)$ to the state (e, f, g, a, b, c) at the constant temperature t is $E-E_0$. But this change in the intrinsic energy is equal to the work W done by the external forces, together with the mechanical value of the heat supplied to the solid during the process.

Thus if J be the mechanical equivalent of the thermal unit

$$W+J.H=E-E_0,$$

or $$W=\phi(e, f, g, a, b, c, t)-\phi(e_0, f_0, g_0, a_0, b_0, c_0, t)$$
$$-J\{\psi(e, f, g, a, b, c, t)-\psi(e_0, f_0, g_0, a_0, b_0, c_0, t)\}\ldots\ldots(24).$$

Hence W is a function of e, f, g, a, b, c, and is independent of the particular succession of states through which the solid passes during the strain at constant temperature.

When the solid is strained in such a manner that no heat is allowed to escape, let t_0 be the initial, and t the final temperature, and suppose that the initial state is taken as the standard. Then the heat supplied during the process is zero and the whole change of intrinsic energy is equal to the work done by the external forces. With our choice of the standard state, this statement is equivalent to the equation

$$W=E=\phi(e, f, g, a, b, c, t)\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

Now the temperature of the solid will be a function of (e, f, g, a, b, c), since the solid is strained according to the adiabatic law, and it follows that W is a function of (e, f, g, a, b, c) and is in this case also independent of the series of intermediate states which can be passed through when no heat is allowed to

pass into or out of any element of the solid. This probably applies to the small vibrations of solid bodies, the period being so small, that no heat is gained or lost during it, and we shall therefore be able to use the energy-function to obtain the equations of vibration.

In any other case W as a function of (e, f, g, a, b, c) does not exist. We have always the relations, given by the general theories of the Conservation and Dissipation of Energy, in the forms

$$\left.\begin{aligned} &\int dW + \int J dH = E - E_0 \\ &E = \phi(e, f, g, a, b, c, t) \\ &\int \frac{dH}{t} = \chi(e, f, g, a, b, c, t) - \chi(e, f, g, a, b, c, t)_0 \end{aligned}\right\} \quad \ldots\ldots(26),$$

where dH is the heat supplied to the solid when its state is changed by infinitely small variations of the quantities (e, f, g, a, b, c, t), and the integrations are summations taken with reference to the series of states through which the solid passes.

It is apparent that, until the form of ϕ is known, we can assert nothing concerning the behaviour of the strained solid, except in the cases when W is known to exist. Thus, if the solid be strained by unequal heating, the theory of elasticity is incapable of answering any question relating to such strain without some additional assumption. Attempts to give an answer have been made by Duhamel and Neumann starting from particular hypotheses. The results at which they arrived could be obtained by assuming that, when the temperature of an element is increased by t, the work done by external forces, in slightly increasing the strain in this element, is

$$\delta\phi(e, f, g, a, b, c) - \beta t \delta(e + f + g) \ldots\ldots\ldots\ldots(27),$$

where ϕ is the same function that would occur if t were constant, and β is a constant coefficient.

64. Green's method[1].

When the function W exists the general variational equation of small motion is

$$+\iiint\left[\left(\rho X - \rho\frac{\partial^2 u}{\partial t^2}\right)\delta u + \left(\rho Y - \rho\frac{\partial^2 v}{\partial t^2}\right)\delta v + \left(\rho Z - \rho\frac{\partial^2 w}{\partial t^2}\right)\delta w\right] dx\,dy\,dz$$
$$+\iint(F\delta u + G\delta v + H\delta w)\,dS - \iiint \delta W\,dx\,dy\,dz = 0 \ldots(28),$$

[1] 'On the Laws of Reflexion and Refraction of Light',... *Mathematical Papers of the late George Green.*

in which ρ is the density, X, Y, Z the components of the bodily force per unit mass at any point (x, y, z), u, v, w the component displacements of the particle initially at (x, y, z), and F, G, H the component surface-tractions per unit area applied at the point of the bounding surface where the element of surface is dS. For this equation represents that the whole work done by all the forces internal and external in a small displacement vanishes.

If we transform this equation by writing

$$\delta W = \frac{\partial W}{\partial e}\delta e + \frac{\partial W}{\partial f}\delta f + \frac{\partial W}{\partial g}\delta g + \frac{\partial W}{\partial a}\delta a + \frac{\partial W}{\partial b}\delta b + \frac{\partial W}{\partial c}\delta c,$$

remember that $$e = \frac{\partial u}{\partial x}, \ldots,$$

and note that $$\iiint \frac{\partial W}{\partial e}\delta\frac{\partial u}{\partial x}\,dx\,dy\,dz$$

$$= \iiint \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial e}\delta u\right)dx\,dy\,dz - \iiint \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial e}\right)\delta u\,dx\,dy\,dz$$

$$= \iint l\frac{\partial W}{\partial e}\delta u\,dS - \iiint \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial e}\right)\delta u\,dx\,dy\,dz \ldots \ldots\ldots\ldots\ldots(29),$$

where l, m, n are the direction-cosines of the normal to dS drawn outwards (from the region occupied by the solid into the space outside), and treat the other terms in like manner, we shall see that the left hand side of equation (28) becomes the sum of a number of volume-integrals and a number of surface-integrals, and these may be arranged so that there is one of each containing δu, one of each containing δv, and one of each containing δw. We must equate to zero the quantities that multiply each of these variations under each of the signs of integration, and we thus obtain

three differential equations that hold at every point of the solid,

and *three equations of condition that hold at all points of the surface.*

65. Form of the Energy Function W[1].

We found, art. 18, equation (19),

$$\delta W = P\delta e + Q\delta f + R\delta g + S\delta a + T\delta b + U\delta c \ldots\ldots (30),$$

so that $$P = \frac{\partial W}{\partial e}, \ldots \quad S = \frac{\partial W}{\partial a}, \ldots \quad \ldots\ldots\ldots\ldots(31).$$

[1] Kirchhoff, *Vorlesungen über mathematische Physik, Mechanik.*

Now, according to the generalised Hooke's Law, the stress-components P, Q, R, S, T, U are linear functions of the strains e, f, g, a, b, c, and therefore W is a quadratic function of the strains. A complete quadratic function of six arguments involves 21 independent coefficients, and therefore there are in general 21 constants of elasticity. As in art. 31 these may be denoted by c's with double suffixes, and then $c_{rs} = c_{sr}$.

We may adopt a symbolical notation. Writing the six strains

$$e = x_1,\ f = x_2,\ g = x_3,\ a = x_4,\ b = x_5,\ c = x_6 \ldots\ldots (32),$$

and writing down the form

$$\Theta = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 \ldots\ldots (33),$$

we may put

$$W = \tfrac{1}{2}\Theta^2 \ldots\ldots (34),$$

where it is understood that $c_1, c_2, \ldots$ have no physical meaning, but after the square of Θ has been formed c_1^2 is to be replaced by c_{11}, c_1c_2 by c_{12} and so on. Then the quantities $c_{11}, c_{12}, \ldots$ are the elastic constants.

Our proof of the existence of W, in the two cases of isothermal and adiabatic changes of state, points to different values of W as a function of (e, f, g, a, b, c) in these two cases. The form of the two functions is the same, since each is a homogeneous quadratic function of the same six arguments, but the coefficients of the two functions may be different. This probably explains some of the discrepancies in the values of the elastic constants as calculated from experiments on equilibrium, and experiments on small vibrations. (cf. Lord Rayleigh, "Sound", ch. XI.)

We shall in our equations express the constants for equilibrium and small vibrations by the same letters. The difference between them is small for all hard solids which have been subjected to experiment.

It may be remarked here that, if a solid be strained at constant temperature, then, whether Hooke's Law hold or not, and whether the strain be small or not, the above equation (28) will still hold good. As soon as we know the stress-strain relations appropriate to the case, it will be possible to proceed to form the equations of motion or equilibrium.

66. Kirchhoff's General Theorems.

Kirchhoff has given some general theorems, founded on the form of W.

(α) Since the solid is supposed disturbed from a state of stable equilibrium, W is always positive. Thus W is a quadratic function of six arguments e, f, g, a, b, c which is always positive.

(β) If $W=0$ the solid is only moved as a rigid body.

For W is a quadratic function of six arguments, which is always positive for all real finite values of these arguments. Thus if the strains be real W can vanish only when the strains all vanish, so that, if $W=0$, we must have

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial v}{\partial y}=0,\ \frac{\partial w}{\partial z}=0,\ \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=0,\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0,\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0 \quad (35).$$

If we differentiate each of these equations with respect to x, y and z we shall find eighteen linear equations among the eighteen second differential coefficients of u, v, w which shew that each of these second differential coefficients must vanish. Thus u, v, w are linear functions of x, y, z, and, in virtue of the above equations, these must be of the forms

$$u=u_0-ry+qz,\ v=v_0-pz+rx,\ w=w_0-qx+py \ldots (36),$$

which are the formulae for the displacement of a rigid body by a translation (u_0, v_0, w_0), and a rotation (p, q, r).

(γ) If the six strains be given, and we seek the displacements u, v, w, then these will be arbitrary to the extent of quantities of the form given in (β). But, if we impose six independent equations of condition, such as that, when $x=0,\ y=0,\ z=0$,

$$u=0,\quad v=0,\quad w=0,\quad \frac{\partial u}{\partial z}=0,\quad \frac{\partial v}{\partial z}=0,\quad \frac{\partial v}{\partial x}=0,$$

the expression for the displacements with given strains will be unique. These particular equations indicate that a point of the solid (the origin) retains its primitive position, that one line-element of the solid (along the axis of z) retains its direction, and that one plane-element of the solid (the zx plane) retains its direction unaltered by the strain. It is manifestly possible, having strained the solid in an arbitrary manner, to bring it back, by translation and rotation, so that this point, this line, and this plane shall recover their primitive positions.

(δ) The strains cannot be given arbitrarily[1].

[1] This theorem was given by Saint-Venant in his edition of Navier's *Leçons*, *Appendice* III.

Suppose we have the equations

$$e=\frac{\partial u}{\partial x},\ f=\frac{\partial v}{\partial y},\ g=\frac{\partial w}{\partial z},\ a=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z},\ b=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\ c=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\quad(37),$$

and seek the conditions of compatibility.

The displacements u, v, w must be one-valued functions of x, y, z, so that, if we integrate along any line drawn through the origin, we must have

$$u=u_0+\int\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz\quad\ldots\ldots\ldots\ldots\ldots\ldots(38),$$

where u_0 is the value of u at the origin.

In like manner we must have

$$\frac{\partial u}{\partial y}=\left(\frac{\partial u}{\partial y}\right)_0+\int\frac{\partial^2 u}{\partial x\partial y}dx+\frac{\partial^2 u}{\partial y^2}dy+\frac{\partial^2 u}{\partial y\partial z}dz\quad\ldots\ldots\ldots\ldots(39),$$

$$\frac{\partial u}{\partial z}=\left(\frac{\partial u}{\partial z}\right)_0+\int\frac{\partial^2 u}{\partial x\partial z}dx+\frac{\partial^2 u}{\partial y\partial z}dy+\frac{\partial^2 u}{\partial z^2}dz\quad\ldots\ldots\ldots\ldots(40),$$

where the line-integrals may be taken along any line leading from the origin to the point (x, y, z), and must be independent of the path of integration.

Now using the equations (37), and re-writing, we may express these in the form

$$\left.\begin{aligned}\frac{\partial u}{\partial y}&=\left(\frac{\partial u}{\partial y}\right)_0+\int\frac{\partial e}{\partial y}dx+\left(\frac{\partial c}{\partial y}-\frac{\partial f}{\partial x}\right)dy+\tfrac{1}{2}\left(\frac{\partial c}{\partial z}+\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\right)dz\\ \frac{\partial u}{\partial z}&=\left(\frac{\partial u}{\partial z}\right)_0+\int\frac{\partial e}{\partial z}dx+\tfrac{1}{2}\left(\frac{\partial c}{\partial z}+\frac{\partial b}{\partial y}-\frac{\partial a}{\partial x}\right)dy+\left(\frac{\partial b}{\partial z}-\frac{\partial g}{\partial x}\right)dz\end{aligned}\right\}\quad\ldots(41).$$

The line integrals will be independent of the path of integration, if

$$\left.\begin{aligned}\frac{\partial^2 e}{\partial y^2}+\frac{\partial^2 f}{\partial x^2}&=\frac{\partial^2 c}{\partial x\partial y}\\ \frac{\partial^2 e}{\partial z^2}+\frac{\partial^2 g}{\partial x^2}&=\frac{\partial^2 b}{\partial z\partial x}\\ 2\frac{\partial^2 e}{\partial y\partial z}+\frac{\partial^2 a}{\partial x^2}&=\frac{\partial^2 b}{\partial x\partial y}+\frac{\partial^2 c}{\partial z\partial x}\\ 2\frac{\partial^2 f}{\partial z\partial x}+\frac{\partial^2 b}{\partial y^2}&=\frac{\partial^2 c}{\partial y\partial z}+\frac{\partial^2 a}{\partial x\partial y}\\ 2\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 c}{\partial z^2}&=\frac{\partial^2 a}{\partial z\partial x}+\frac{\partial^2 b}{\partial y\partial z}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(42),$$

to which may be added

$$\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 g}{\partial y^2} = \frac{\partial^2 a}{\partial y \partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43),$$

one of the equations obtained by operating in like manner upon v.

These six equations are the necessary and sufficient conditions of compatibility of equations (37).

(ϵ) If either the surface-displacements or the surface-tractions be given, the solution of the general equations of equilibrium is unique.

1°. Supposing the bodily forces and surface-tractions given, then, taking W a quadratic function of the six strains, we have

$$\frac{\partial W}{\partial e} = P, \ldots\ldots$$

also the general equations of equilibrium are three such as

$$\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} + \rho X = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (44),$$

and the boundary-conditions are three such as

$$lP + mU + nT = F \quad \ldots\ldots\ldots\ldots\ldots\ldots (45).$$

If possible suppose there are two different solutions of these sets of equations, and that the corresponding displacements are u_1, v_1, w_1 and u_2, v_2, w_2 in the two solutions. Then, writing

$$u' = u_1 - u_2, \quad v' = v_1 - v_2, \quad w' = w_1 - w_2,$$

we see that u', v', w' are a set of displacements which satisfy three such differential equations as

$$\frac{\partial P'}{\partial x} + \frac{\partial U'}{\partial y} + \frac{\partial T'}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (46),$$

and three such boundary-conditions as

$$lP' + mU' + nT' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (47),$$

where $P', \ldots$ are the stresses corresponding to these displacements.

Now, by Green's transformation,

$$\iiint \left\{ u' \left(\frac{\partial P'}{\partial x} + \frac{\partial U'}{\partial y} + \frac{\partial T'}{\partial z} \right) + \ldots + \ldots \right\} dx dy dz$$

$$= \iint \{ u' (lP' + mU' + nT') + \ldots + \ldots \} \, dS$$

$$- \iiint \left(e' \frac{\partial W'}{\partial e'} + f' \frac{\partial W'}{\partial f'} + \ldots + c' \frac{\partial W'}{\partial c'} \right) dx dy dz,$$

where e', f',...c' are the strains corresponding to the displacements u', v', w'. Hence

$$\iiint\left(e'\frac{\partial W'}{\partial e'}+\ldots+c'\frac{\partial W'}{\partial c'}\right)dxdydz=0 \ldots\ldots\ldots(48).$$

But, from the form of W', as a positive quadratic function, we know that the expression under the integral sign is $2W'$, so that the integral is a sum of positive terms, which can vanish only when $e'=f'=\ldots=c'=0$. Thus the displacements (u', v', w') are such as are possible for a rigid body, and the solution is only indeterminate to the extent of such displacements.

2°. Supposing the bodily forces and surface-displacements given, we take as before two solutions u_1, v_1, w_1, u_2, v_2, w_2, and form their differences u', v', w', then u', v', w' satisfy stress-equations like

$$\frac{\partial P'}{\partial x}+\frac{\partial U'}{\partial y}+\frac{\partial T'}{\partial z}=0,$$

and boundary-conditions $u'=0$, $v'=0$, $w'=0$, at the surface.

Thus we find that

$$\iiint\left(e'\frac{\partial W}{\partial e'}+\ldots+c'\frac{\partial W}{\partial e'}\right)dxdydz=0,$$

and hence $e'=0$, $f'=0$,...$c'=0$, and the displacements are only indeterminate to the extent of displacements possible for a rigid body. This indeterminateness is also removed, since u, v, w are given at the surface, and, if three points of a rigid body be moved in a given manner, the displacement of all the points is determinate.

It follows from this theorem that, if in any manner we can find a solution of the equations of equilibrium, which satisfies all the conditions, this is the only solution.

There are exceptional cases, in which more than one solution is possible; in these cases one of the solutions that can be obtained corresponds to an unstable condition[1]. It will be observed that, if we had set out from the variational equation, the stability of the system would have depended upon second variations. It will be better to postpone the general consideration of the theory of stability of elastic systems until after we have discussed the theory

[1] Sir W. Thomson, *Math. and Phys. Papers*, vol. III., G. H. Bryan, *Camb. Phil. Soc. Proc.*, vol. VI. 1888.

of the deformation of thin wires and plates, for which possible instability becomes of importance. Kirchhoff's theorem of the uniqueness of solutions holds for the cases we shall investigate in this volume, viz. those of infinitesimal strain in a body all whose dimensions are finite.

67. Possibility of Solution.

We add here a theorem relating to the possibility of solving the general equations.

Suppose the surface-displacements given, and that there is no bodily force, we have to shew that there exist functions u, v, w, satisfying the differential equations of equilibrium and the boundary-conditions.

Let u, v, w and u', v', w' be two systems of functions of x, y, z which are finite, continuous, and one-valued within the surface S, limiting the body, and at that surface become equal to the given displacements. It is clear that there is an infinite number of such systems of functions. Form with u, v, w the energy-function W, which is a definite quadratic function of the six strains, expressed by certain differential coefficients of u, v, w; and form in like manner with u', v', w' the energy-function W', and let

$$V = \iiint W\,dx\,dy\,dz, \qquad V' = \iiint W'\,dx\,dy\,dz \ldots\ldots\ldots(49),$$

the integration extending throughout the body, then

$$V' - V = \iiint (W' - W)\,dx\,dy\,dz \ldots\ldots\ldots(50).$$

Denote the six strains corresponding to u, v, w by the letters $e_1, e_2, \ldots e_6$, and the corresponding six stresses by $P_1, P_2, \ldots P_6$, and let similar quantities derived from the u', v', w' system be denoted by the same letters with accents, and write

$$W = f(e_1, e_2, \ldots e_6) = f(e)\ldots\ldots\ldots\ldots\ldots\ldots(51),$$

then with the same notation

$$W' = f\{e + (e' - e)\}\ldots\ldots\ldots\ldots\ldots\ldots(52),$$

and
$$V' - V = \iiint \{f\{e + (e' - e)\} - f(e)\}\,dx\,dy\,dz$$
$$= \iiint \left\{\Sigma (e' - e)\frac{\partial f(e)}{\partial e} + f(e' - e)\right\} dx\,dy\,dz \ldots. (53),$$

since f is a homogeneous quadratic function of its six arguments.

Then it is easy to shew that the right hand side of (53) can be transformed into

$$-\iiint (u'-u)\left(\frac{\partial P_1}{\partial x}+\frac{\partial P_6}{\partial y}+\frac{\partial P_5}{\partial z}\right)dxdydz$$

$-$(similar terms in $v'-v$ and $w'-w$)$+\iiint f(e'-e)\,dxdydz$...(54),

and therefore, since the function $f(e'-e)$ is necessarily positive, it follows from (53) that V' is always greater than V, if the equation

$$\frac{\partial P_1}{\partial x}+\frac{\partial P_6}{\partial y}+\frac{\partial P_5}{\partial z}=0 \quad \text{.....(55),}$$

and the two similar equations hold good, and therefore these equations have a solution if V have a minimum value.

Now W is a sum of positive terms, and, at the surface, W must be different from zero, and therefore V being necessarily positive must have a minimum value for some set of functions u, v, w satisfying the boundary-conditions. This shews that the equations such as (55) have a solution, and that the values of u, v, w thence deducible make V a minimum. Since these are the equations of equilibrium, the theorem is proved.

This is an expansion of the proof sketched by Sir William Thomson (*Math. and Phys. Papers*, vol. III., p. 389). It is similar to the well-known "proof" usually given of Dirichlet's *Principle*. This proof has been attacked by Weierstrass, and is not admitted as valid in its application to the Theory of Functions of a complex variable. The difficulty appears to be one concerning the continuity of the integral; it is not shewn that the minimum value, proved to exist, can be arrived at by continuous variation from a function V', otherwise arbitrary, but satisfying the boundary-conditions. The difficulty has not yet been cleared up.

68. Betti's Theorem.

Let u, v, w be any functions of x, y, z, t which are finite, continuous, and one-valued within the space occupied by a solid. Then, if proper bodily forces and surface-tractions be applied to the solid, u, v, w can be the component displacements of a point of it. These bodily forces and surface-tractions can be determined from the equations of equilibrium or small motion, and, when they are known, u, v, w will be the system of displacements that such bodily forces and surface-tractions would produce.

Now let (u_1, v_1, w_1), (u_2, v_2, w_2) be two sets of displacements, (X_1, Y_1, Z_1), and (X_2, Y_2, Z_2) the corresponding bodily forces, (F_1, G_1, H_1), and (F_2, G_2, H_2) the corresponding surface-tractions, then Prof. Betti's theorem is that *the whole work done by all the forces (including kinetic reactions) of the first set, acting over the displacements produced by the second set, is equal to the whole work done by the forces of the second set, acting over the displacements produced by the first.*

Analytically stated this gives the equation

$$\iiint\left[\rho\left(X_1-\frac{\partial^2 u_1}{\partial t^2}\right)u_2+\rho\left(Y_1-\frac{\partial^2 v_1}{\partial t^2}\right)v_2+\rho\left(Z_1-\frac{\partial^2 w_1}{\partial t^2}\right)w_2\right]dxdydz$$
$$+\iint(F_1u_2+G_1v_2+H_1w_2)\,dS$$
$$=\iiint\left[\rho\left(X_2-\frac{\partial^2 u_2}{\partial t^2}\right)u_1+\rho\left(Y_2-\frac{\partial^2 v_2}{\partial t^2}\right)v_1+\rho\left(Z_2-\frac{\partial^2 w_2}{\partial t^2}\right)w_1\right]dxdydz$$
$$+\iint(F_2u_1+G_2v_1+H_2w_1)\,dS\ldots\ldots(56),$$

the volume integrations being taken over all points within the surface S of the solid.

To prove the theorem, let P_1, Q_1, R_1, S_1, T_1, U_1 be the six stresses, and $e_1, f_1, g_1, a_1, b_1, c_1$ the six strains, corresponding to the first set of displacements, and like expressions with suffix 2 those corresponding to the second set, and let W_1 be the energy-function for the first set, and W_2 that for the second set. Then, by the equations of equilibrium, the left hand side of (56) is

$$-\iiint\left[\left(\frac{\partial P_1}{\partial x}+\frac{\partial U_1}{\partial y}+\frac{\partial T_1}{\partial z}\right)u_2+\left(\frac{\partial U_1}{\partial x}+\frac{\partial Q_1}{\partial y}+\frac{\partial S_1}{\partial z}\right)v_2\right.$$
$$\left.+\left(\frac{\partial T_1}{\partial x}+\frac{\partial S_1}{\partial y}+\frac{\partial R_1}{\partial z}\right)w_2\right]dxdydz+\iint(F_1u_2+G_1v_2+H_1w_2)\,dS$$
$$=\iiint(P_1e_2+Q_1f_2+R_1g_2+S_1a_2+T_1b_2+U_1c_2)\,dxdydz$$
$$-\iint[u_2(lP_1+mU_1+nT_1-F_1)+v_2(lU_1+mQ_1+nS_1-G_1)$$
$$+w_2(lT_1+mS_1+nR_1-H_1)]\,dS.$$

The surface integral vanishes identically in virtue of the boundary-conditions, and the volume integral is

$$\iiint\left[e_2\frac{\partial W_1}{\partial e_1}+f_2\frac{\partial W_1}{\partial f_1}+g_2\frac{\partial W_1}{\partial g_1}+a_2\frac{\partial W_1}{\partial a_1}+b_2\frac{\partial W_1}{\partial b_1}+c_2\frac{\partial W_1}{\partial c_1}\right]dxdydz,$$

and this, by a property of quadratic functions, is equal to

$$\iiint\left[e_1\frac{\partial W_2}{\partial e_2}+f_1\frac{\partial W_2}{\partial f_2}+g_1\frac{\partial W_2}{\partial g_2}+a_1\frac{\partial W_2}{\partial a_2}+b_1\frac{\partial W_2}{\partial b_2}+c_1\frac{\partial W_2}{\partial c_2}\right]dxdydz,$$

which, as before, is equal to the right hand member of (56).

This theorem[1] is equivalent in the subject of Elasticity to Lord Rayleigh's general reciprocal theorem for any system displaced from a configuration of stable equilibrium. Its utility, as the foundation of a method of integration of the general equations, will be hereafter explained.

69. Determination of the mean values of strain-components.

Prof. Betti has given an interesting example of the use of his theorem, shewing how the mean values of the strains produced in a body of any form by given surface-tractions can be calculated.

Let the system of displacements u_2, v_2, w_2 correspond to a homogeneous pure strain given by the equations

$$\left.\begin{aligned} u_2 &= e_2x + \tfrac{1}{2}c_2y + \tfrac{1}{2}b_2z \\ v_2 &= \tfrac{1}{2}c_2x + f_2y + \tfrac{1}{2}a_2z \\ w_2 &= \tfrac{1}{2}b_2x + \tfrac{1}{2}a_2y + g_2z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (57),$$

and let the system u_1, v_1, w_1 be produced by given surface-tractions F_1, G_1, H_1. Then the theorem for displacements produced by surface-tractions becomes

$$\iint(F_1u_2+G_1v_2+H_1w_2)\,dS=\iint(F_2u_1+G_2v_1+H_2w_1)\,dS \ldots (58).$$

The six stresses P_2, Q_2, R_2, S_2, T_2, U_2 are all constants, and thus, substituting

$$F_2=lP_2+mU_2+nT_2,\quad G_2=lU_2+mQ_2+nS_2,\quad H_2=lT_2+mS_2+nR_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (59),$$

the equation becomes by transformation of the right hand surface-integral into a volume-integral

$$\begin{aligned}\iint[F_1e_2x+G_1f_2y+H_1g_2z+\tfrac{1}{2}a_2(H_1y+G_1z)+\tfrac{1}{2}b_2(F_1z+H_1x) \\ +\tfrac{1}{2}c_2(G_1x+F_1y)]\,dS=P_2\iiint e_1dV+Q_2\iiint f_1dV+R_2\iiint g_1dV \\ +S_2\iiint a_1dV+T_2\iiint b_1dV+U_2\iiint c_1dV \quad \ldots\ldots\ldots\ldots\ldots (60),\end{aligned}$$

where dV is the element of volume.

[1] Betti's theorem was first given in *Il Nuovo Cimento*, 1872. Its publication precedes by one year that of Lord Rayleigh's reciprocal theorem.

Now it is easy, by solving the stress-strain equations, to find e_2, f_2, g_2, a_2, b_2, c_2 so that one of the stresses P_2,... is unity and the rest zero, and thus from the above equation we can find the volume-integrals of each of the six strain-components.

Consider, in particular, the case of an isotropic solid, for which

$$\left.\begin{aligned} P_2 &= (\lambda + 2\mu)\, e_2 + \lambda\,(f_2 + g_2) \\ &\cdots\cdots \\ S_2 &= \mu a_2 \\ &\cdots\cdots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(61).$$

In order that P_2 may be unity and the rest zero, we must have

$$e_2 = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} = \frac{1}{E},\quad f_2 = g_2 = -\frac{\lambda}{2\mu(3\lambda+2\mu)} = \frac{-\sigma}{E},\quad a_2 = b_2 = c_2 = 0.$$

In order that S_2 may be unity and the rest zero, we must have

$$a_2 = \frac{1}{\mu},\quad e_2 = f_2 = g_2 = 0,\quad b_2 = c_2 = 0.$$

Hence, suppressing the suffix 1, we find the mean values of the six strains in the following forms:

$$\text{mean value of}\quad e = \frac{1}{EV}\iint\{Fx - \sigma\,(Gy + Hz)\}\,dS \ldots\ldots\ldots\ldots(62),$$

$$\text{mean value of}\quad a = \frac{1}{2\mu V}\iint(Hy + Gz)\,dS \ldots\ldots\ldots\ldots\ldots\ldots(63),$$

where V is the volume of the body, and E is the Young's Modulus, μ the Rigidity, and σ the Poisson's ratio of the material, and these three constants are connected by the relation

$$E = 2\mu\,(1+\sigma).$$

We notice that the mean value of the cubical compression is

$$\text{mean value of}\quad \Delta = \frac{1}{3kV}\iint(Fx + Gy + Hz)\,dS \ldots\ldots\ldots\ldots(64),$$

where k is the resistance to compression, given by

$$k = \lambda + \tfrac{2}{3}\mu = \tfrac{2}{3}\mu\,\frac{1+\sigma}{1-2\sigma}.$$

THE PROPAGATION OF A DISTURBANCE IN AN ISOTROPIC ELASTIC MEDIUM[1].

70. Solution of the equations of small motion.

We shall suppose a medium unlimited in extent. At any instant, which is taken as the origin of time, we shall suppose the medium disturbed, so that different parts of it are in different states as regards compression and distortion, and we seek to find the state of the medium at any time t, at any point. We begin with the simple case of a homogeneous isotropic medium.

The differential equations of small motion of the medium are

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u&=\rho\frac{\partial^2 u}{\partial t^2}\\(\lambda+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 v&=\rho\frac{\partial^2 v}{\partial t^2}\\(\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w&=\rho\frac{\partial^2 w}{\partial t^2}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(65).$$

By differentiating with respect to x, y, z, and adding, we obtain the equation

$$(\lambda+2\mu)\nabla^2\Delta=\rho\frac{\partial^2\Delta}{\partial t^2}\quad\ldots\ldots\ldots\ldots\ldots(66).$$

By differentiating the third of equations (65) with respect to y and the second with respect to z, and subtracting, we obtain the equation

$$\mu\nabla^2\varpi_1=\rho\frac{\partial^2\varpi_1}{\partial t^2}\quad\ldots\ldots\ldots\ldots\ldots(67),$$

where

$$\varpi_1=\tfrac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

is the rotation about an axis parallel to the axis x, and in like manner

$$\left.\begin{aligned}\mu\nabla^2\varpi_2&=\rho\frac{\partial^2\varpi_2}{\partial t^2}\\\mu\nabla^2\varpi_3&=\rho\frac{\partial^2\varpi_3}{\partial t^2}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(68).$$

[1] See Poisson's memoir of April 14, 1828, *Mém. Paris Acad.* 1829, and Lord Rayleigh's *Theory of Sound*, vol. II. ch. XIV.

These equations, with the identical equation

$$\frac{\partial \varpi_1}{\partial x} + \frac{\partial \varpi_2}{\partial y} + \frac{\partial \varpi_3}{\partial z} = 0 \quad \text{.....(69),}$$

determine the whole motion.

In what follows we shall for shortness write

$$(\lambda + 2\mu) = h^2\rho, \quad \mu = k^2\rho \quad \text{.....(70).}$$

We have now to consider the solution of the equations

$$\left.\begin{aligned} h^2\nabla^2\Delta &= \frac{\partial^2\Delta}{\partial t^2} \\ k^2\nabla^2\varpi &= \frac{\partial^2\varpi}{\partial t^2}, \quad \varpi = (\varpi_1, \varpi_2, \varpi_3) \end{aligned}\right\} \quad \text{.....(71).}$$

These equations are of the same type

$$\frac{\partial^2\phi}{\partial t^2} = a^2\nabla^2\phi \quad \text{.....(72),}$$

and we want a solution of this in terms of arbitrary initial conditions.

We can write down at once the symbolical solution

$$\phi = \cosh(a\nabla t)\,\phi_1 + t\,\frac{\sinh(a\nabla t)}{a\nabla t}\,\phi_2 \quad \text{.....(73),}$$

from which it appears that initially

$$\left.\begin{aligned} \phi &= \phi_1 \\ \frac{\partial\phi}{\partial t} &= \phi_2 \end{aligned}\right\} \quad \text{.....(74),}$$

so that, if the initial values of ϕ and $\frac{\partial\phi}{\partial t}$ be denoted by ϕ_0 and $\dot{\phi}_0$,

$$\phi = \cosh(at\nabla)\,\phi_0 + t\,\frac{\sinh(at\nabla)}{at\nabla}\,\dot{\phi}_0 \quad \text{.....(75),}$$

where ϕ_0, $\dot{\phi}_0$ are functions of x, y, z.

Observing that $\cosh(at\nabla)$, and $\frac{\sinh(at\nabla)}{at\nabla}$ are even functions of ∇, we see that these are real operators, and the operations indicated can be performed.

But there is another form into which the solution can be thrown, in virtue of the theorem that the mean value of a function ψ over the surface of a sphere of radius R, whose centre is the origin, is the value of $\frac{\sinh(R\nabla)}{R\nabla}\psi$ at the origin.

For, consider the function $\iint e^{ax+by+cz}dS$, the integration extending over the surface of a sphere, whose centre is the origin, and whose radius is R. Changing the axes, so that the new axis of Z may be the normal to the plane $ax + by + cz = 0$, we see that

$$Z\sqrt{a^2+b^2+c^2} = ax + by + cz,$$

also $dS = 2\pi R dZ$, and the integration for Z is taken between the limits R and $-R$.

Hence
$$\iint e^{ax+by+cz}dS = \int_{-R}^{R} 2\pi R e^{\sqrt{a^2+b^2+c^2}Z}dZ$$

$$= \frac{4\pi R^2}{R\nabla}\sinh R\nabla,$$

if
$$\nabla^2 = a^2 + b^2 + c^2.$$

Now suppose ψ is any function of x, y, z, uniform within a sphere whose centre is the origin and radius R,

then
$$\psi(x, y, z) = e^{\left(x\frac{d}{dx_0}+y\frac{d}{dy_0}+z\frac{d}{dz_0}\right)}\psi(x_0, y_0, z_0),$$

by Maclaurin's theorem, where x_0, y_0, z_0 are to be put equal to zero after the differentiations have been performed.

Hence the mean value of the function over the surface of the sphere

$$= \frac{1}{4\pi R^2}\iint \psi(x, y, z)\, dS$$

$$= \frac{1}{4\pi R^2}\iint e^{x\frac{d}{dx_0}+y\frac{d}{dy_0}+z\frac{d}{dz_0}}dS\ \ \psi(x_0, y_0, z_0)$$

$$= \frac{\sinh(R\nabla_0)}{R\nabla_0}\psi(x_0, y_0, z_0),$$

where
$$\nabla_0^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + \frac{\partial^2}{\partial z_0^2},$$

and x_0, y_0, z_0 are to be put equal to zero after the differentiation: this proves the theorem.

Now
$$\cosh(at\nabla) = \frac{\partial}{\partial t}\left(t\frac{\sinh(at\nabla)}{at\nabla}\right).$$

Hence the general solution of the equation

$$\frac{\partial^2\phi}{\partial t^2} = a^2\nabla^2\phi$$

in terms of initial conditions is shewn to be

$$\phi = \frac{\partial}{\partial t}\left\{t\frac{\sinh(at\nabla)}{at\nabla}\phi_0\right\} + t\frac{\sinh(at\nabla)}{at\nabla}\dot{\phi}_0 \quad \ldots\ldots\ldots (76),$$

and this solution can be interpreted as follows:

Take any point of the medium as origin, and with this point as centre describe a sphere of radius at, then the function

$$\frac{\sinh(at\nabla)}{at\nabla}\dot{\phi}_0$$

is the mean value of $\dot{\phi}_0$ over the surface of this sphere, and

$$\frac{\sinh(at\nabla)}{at\nabla}\phi_0$$

is the mean value of ϕ_0 over the surface of this sphere, thus

$$\phi = \frac{\partial}{\partial t}(t\bar{\phi}_0) + t\dot{\bar{\phi}}_0 \quad \ldots\ldots\ldots\ldots\ldots (77),$$

where $\bar{\phi}_0$ and $\dot{\bar{\phi}}_0$ are the mean values of the initial ϕ and $\dot{\phi}$ at all points of the surface of a sphere whose radius is at and centre the point at which the disturbance is to be estimated.

71. Interpretation. Wave-motion[1].

Now suppose the initial disturbance confined within a certain space T. Then at time $t = 0$ all the medium without the surface of T is at rest, and ϕ_0 $\dot{\phi}_0$ have values different from zero for points within T, and are zero outside. With any point of T as centre describe a sphere of radius at, then at time t the disturbance will be confined to the space within the envelope of these spheres. This envelope is a surface of two sheets, an inner and an outer, and the part of the medium between the two sheets is in motion, all the remainder is at rest. Each element of the medium as the outer sheet of the envelope reaches it takes suddenly the small velocity corresponding to $\dot{\phi}$, and after the inner sheet passes it suddenly loses velocity and comes to rest. This kind of motion is called wave-motion. If the disturbance emanate from the space close about a central point it is clear that there will be at any instant two concentric spheres very close together whose common centre is at the point, and the disturbed parts of the medium will be those between the two spheres. The radius of the mean sphere

[1] Stokes, 'Dynamical Theory of Diffraction', *Math. and Phys. Papers*, vol. II.

at time t will be at. The waves are therefore said to be propagated with velocity a.

In the case of the isotropic solid, we have two kinds of waves. The first is a wave of compression corresponding to equation (66), and travelling with a velocity $h = \sqrt{(\lambda + 2\mu)/\rho}$, the other kind are waves of distortion, corresponding to equations (67) and (68), and travelling with a velocity $k = \sqrt{\mu/\rho}$.

72. Propagation of plane waves.

Now suppose that plane waves are propagated through the medium. Then we must have the displacement the same at all points of a certain family of parallel planes, and we may take

$$\left.\begin{aligned} u &= Af(ax + by + cz + et) \\ v &= Bf(ax + by + cz + et) \\ w &= Cf(ax + by + cz + et) \end{aligned}\right\}.$$

The general equations are satisfied by supposing

$$\left.\begin{aligned} \{(\lambda+\mu)a^2 + \mu(a^2+b^2+c^2) - \rho e^2\}A + (\lambda+\mu)abB + (\lambda+\mu)acC &= 0 \\ (\lambda+\mu)abA + \{(\lambda+\mu)b^2 + \mu(a^2+b^2+c^2) - \rho e^2\}B + (\lambda+\mu)bcC &= 0 \\ (\lambda+\mu)acA + (\lambda+\mu)bcB + \{(\lambda+\mu)c^2 + \mu(a^2+b^2+c^2) - \rho e^2\}C &= 0 \end{aligned}\right\}.$$

Let $e^2 = V^2(a^2 + b^2 + c^2)$, then V is the velocity of the waves, and we have, on eliminating A, B, C, an equation which turns out to be

$$(\lambda + 2\mu - \rho V^2)(\mu - \rho V^2)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (78),$$

which gives the values of V, $V = \sqrt{(\lambda + 2\mu)/\rho}$, and $V = \sqrt{\mu/\rho}$, corresponding to waves of compression and to waves of distortion respectively.

PROPAGATION OF A DISTURBANCE IN AN ÆOLOTROPIC MEDIUM.

73. Formation of equations of motion when there is a surface of discontinuity.

The particular case of an isotropic medium, in which the part within a space T is initially compressed and distorted, and the remainder of the medium in its natural state, is included in the more general problem presented by a medium within which there is, at time $t = 0$, a surface of discontinuity S_0. On one side of S_0, which we shall call the positive side, the medium is strained in such a way that the component displacements u, v, w are continuous

functions of the coordinates (x, y, z), and on the other side of S_0, which we shall call the negative side, the displacements are different continuous functions of the coordinates. The difference between two components of displacement on opposite sides of S_0 is zero, the difference of their differential coefficients with respect to x, y, z or t is taken to be of the same order of magnitude as these differential coefficients. We shall shew that the surface of discontinuity is propagated through the medium in such a way that any tangent plane moves parallel to itself, with a velocity depending on its direction and independent of the time. The theory was given by Herr Christoffel in Brioschi's *Annali di Matematica*, 1877.

Suppose then that, at time t, there is in the medium a surface of discontinuity S. On the positive side of S let the displacement be u_1, v_1, w_1 and on the negative side of S let the displacement be u_2, v_2, w_2, then these agree at the surface, but their differential coefficients are different on the two sides. We suppose the tangent plane at any point on S to move in time dt through a small space ωdt with velocity ω in the positive direction of the normal to S, then, in the neighbourhood of the point of contact, a small cylindrical element $\rho\omega dt dS$ of the medium will have its velocity changed from $\dot{u}_1, \dot{v}_1, \dot{w}_1$ to $\dot{u}_2, \dot{v}_2, \dot{w}_2$, and will therefore have been acted upon by an impulse whose components are

$$\left.\begin{aligned} &\rho\omega dt dS\,(\dot{u}_2 - \dot{u}_1) \text{ parallel to } x \\ &\rho\omega dt dS\,(\dot{v}_2 - \dot{v}_1) \text{ parallel to } y \\ &\rho\omega dt dS\,(\dot{w}_2 - \dot{w}_1) \text{ parallel to } z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(79).$$

Now let l, m, n be the direction-cosines of the normal to dS drawn in the positive direction, and let F_1, G_1, H_1 be the surface-tractions on the positive face of the small cylindrical element, F_2, G_2, H_2 those on the negative face, then the impulses of these forces during the time dt are the impulses that change the motion of the element. Hence we have

$$-\rho\omega dS\,(\dot{u}_1 - \dot{u}_2) = (F_1 - F_2)\,dS \quad \ldots\ldots\ldots\ldots(80),$$

and two similar equations.

Also we have, by the ordinary stress-equations,

$$\left.\begin{aligned} F_1 &= lP_1 + mU_1 + nT_1 \\ G_1 &= lU_1 + mQ_1 + nS_1 \\ H_1 &= lT_1 + mS_1 + nR_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(81),$$

and similar equations with suffix 2.

Thus, if for shortness we write

$$P_1 - P_2 = P' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(82),$$

also $$\dot{u}_1 - \dot{u}_2 = \xi, \quad \dot{v}_1 - \dot{v}_2 = \eta, \quad \dot{w}_1 - \dot{w}_2 = \zeta \quad \ldots\ldots\ldots(83),$$

we get

$$\left.\begin{aligned} -\rho\omega\xi &= lP' + mU' + nT' \\ -\rho\omega\eta &= lU' + mQ' + nS' \\ -\rho\omega\zeta &= lT' + mS' + nR' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(84).$$

Now let W be the potential energy of strain, and, as in art. 65, let $x_1, x_2 \ldots x_6$ be the six strains, and write symbolically

$$X = c_1x_1 + c_2x_2 + \ldots + c_6x_6 \quad \ldots\ldots\ldots\ldots\ldots(85),$$

then, symbolically, $$W = \tfrac{1}{2}X^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(86).$$

Let the excess of the strains $x_1, \ldots$ on the positive side of the surface of discontinuity above those on the negative side be denoted by $x_1', x_2', \ldots$, and write down the form

$$X' = c_1x_1' + c_2x_2' + \ldots + c_6x_6' \quad \ldots\ldots\ldots\ldots\ldots(87),$$

and let

$$\begin{aligned} W' &= \tfrac{1}{2}X'^2 \\ &= \tfrac{1}{2}(c_{11}, c_{22} \ldots c_{12} \chi x_1' \ldots x_6')^2 \ldots\ldots\ldots\ldots\ldots(88), \end{aligned}$$

then for any x, $\dfrac{\partial W'}{\partial x'}$ is the same linear function of the quantities x' that $\dfrac{\partial W}{\partial x}$ is of the quantities x, *i.e.* it is the excess of a stress-component on the positive side above the corresponding stress-component on the negative side. Thus the equations (84) become three such as

$$-\rho\omega\xi = l\frac{\partial W'}{\partial x_1'} + m\frac{\partial W'}{\partial x_6'} + n\frac{\partial W'}{\partial x_5'} \quad \ldots\ldots\ldots\ldots(89).$$

74. Conditions at the separating surface.

Now let (α, β, γ) be any point which moves so as always to be on the surface of discontinuity at time t, and write the equation of the surface

$$t = f(\alpha, \beta, \gamma) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(90).$$

Then it is clear that

$$\omega dt = ld\alpha + md\beta + nd\gamma \quad \ldots\ldots\ldots\ldots\ldots(91),$$

and therefore $$\frac{\partial t}{\partial \alpha} = \frac{l}{\omega}, \quad \frac{\partial t}{\partial \beta} = \frac{m}{\omega}, \quad \frac{\partial t}{\partial \gamma} = \frac{n}{\omega} \quad \ldots\ldots\ldots\ldots(92).$$

The equations of continuity of displacement hold at points

(x, y, z) which move so as to remain on the surface, *i.e.* so as to coincide with α, β, γ. Hence we may differentiate the equations

$$u_1 = u_2, \quad v_1 = v_2, \quad w_1 = w_2 \ldots\ldots\ldots\ldots\ldots\ldots (93),$$

with respect to α, β or γ, regarding t as a function of these quantities, and replace the partial differential coefficients with respect to (α, β, γ) by partial differential coefficients with respect to (x, y, z). Doing this we get nine such equations as

$$\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} + (\dot{u}_1 - \dot{u}_2)\frac{\partial t}{\partial \alpha} = 0 \quad \ldots\ldots\ldots\ldots\ldots (94).$$

With notations already introduced we thus obtain

$$\left.\begin{array}{lll} \omega x_1' + l\xi = 0, & \omega x_2' + m\eta = 0, & \omega x_3' + n\zeta = 0 \\ \omega x_4' + n\eta + m\zeta = 0, & \omega x_5' + l\zeta + n\xi = 0, & \omega x_6' + m\xi + l\eta = 0 \end{array}\right\} \quad (95).$$

75. Formation of the equation for the velocity.

Let Π be the function into which W' is transformed by substituting for $x_1', x_2' \ldots x_6'$ from the equations (95). Then Π is a quadratic function of ξ, η, ζ, and, since ξ only occurs in the expressions for x_1', x_5', x_6', we have

$$\frac{\partial \Pi}{\partial \xi} = -\frac{1}{\omega}\left(l\frac{\partial W'}{\partial x_1'} + m\frac{\partial W'}{\partial x_6'} + n\frac{\partial W'}{\partial x_5'}\right) \ldots\ldots\ldots\ldots (96),$$

and similar equations for

$$\frac{\partial \Pi}{\partial \eta}, \quad \frac{\partial \Pi}{\partial \zeta}.$$

And the equations (90) therefore become

$$\rho\xi = \frac{\partial \Pi}{\partial \xi}, \quad \rho\eta = \frac{\partial \Pi}{\partial \eta}, \quad \rho\zeta = \frac{\partial \Pi}{\partial \zeta} \ldots\ldots\ldots\ldots (97).$$

To form the function Π, write down the symbolical equations

$$\left.\begin{array}{l} c_1 l + c_6 m + c_5 n = \lambda_1 \\ c_6 l + c_2 m + c_4 n = \lambda_2 \\ c_5 l + c_4 m + c_3 n = \lambda_3 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (98).$$

Then

$$\omega\,(c_1 x_1' + c_2 x_2' + \ldots + c_6 x_6') = -(\lambda_1\xi + \lambda_2\eta + \lambda_3\zeta) \ldots\ldots (99),$$

and therefore

$$\begin{aligned} \omega^2\Pi &= \tfrac{1}{2}(\lambda_1\xi + \lambda_2\eta + \lambda_3\zeta)^2 \\ &= \tfrac{1}{2}(\lambda_{11}, \lambda_{22}, \lambda_{33}, \lambda_{23}, \lambda_{31}, \lambda_{12}\chi\xi, \eta, \zeta)^2 \quad \ldots\ldots\ldots\ldots (100), \end{aligned}$$

where

$$\lambda_{11}=\lambda_1^2=(c_1l+c_6m+c_5n)^2=(c_{11}, c_{66}, c_{55}, c_{56}, c_{15}, c_{61}\mathcal{X}l, m, n)^2 \ldots (101),$$

and
$$\lambda_{12}=\lambda_1\lambda_2=(c_1l+c_6m+c_5n)(c_6l+c_2m+c_4n)$$
$$=(c_{16}, c_{26}, c_{54}, \tfrac{1}{2}[c_{25}+c_{46}], \tfrac{1}{2}[c_{14}+c_{56}], \tfrac{1}{2}[c_{12}+c_{66}]\mathcal{X}l, m, n)^2 \quad \ldots\ldots\ldots (102),$$

and the other coefficients can be written down in like manner.

The function Π is thus a complete quadratic function of ξ, η, ζ, and, since W' is always positive, $\omega^2\Pi$ also is always positive.

Now from the equations (97) we find

$$\rho\omega^2\xi=\omega^2\frac{\partial\Pi}{\partial\xi}=\lambda_{11}\xi+\lambda_{12}\eta+\lambda_{13}\zeta,$$

and two similar equations.

Hence ω^2 must satisfy the determinantal equation

$$\begin{vmatrix} \lambda_{11}-\omega^2\rho, & \lambda_{12}, & \lambda_{13} \\ \lambda_{12}, & \lambda_{22}-\omega^2\rho, & \lambda_{23} \\ \lambda_{13}, & \lambda_{23}, & \lambda_{33}-\omega^2\rho \end{vmatrix} = 0 \ldots\ldots (103).$$

Since the function $\omega^2\Pi$ is always positive, the roots of this equation are all real and positive.

Thus there are in general three real values of ω, the velocity with which the tangent plane to the surface of discontinuity advances, and these are functions of (l, m, n) the direction of the tangent plane.

76. Equations of a ray.

Let α_0, β_0, γ_0 be any point on the surface S_0 when $t=0$, then the parallel tangent plane at time t is

$$l(x-\alpha_0)+m(y-\beta_0)+n(z-\gamma_0)=\omega t,$$

and this contains the point

$$\left(\alpha_0+t\frac{\partial\omega}{\partial l}, \quad \beta_0+t\frac{\partial\omega}{\partial m}, \quad \gamma_0+t\frac{\partial\omega}{\partial n}\right),$$

if
$$l\frac{\partial\omega}{\partial l}+m\frac{\partial\omega}{\partial m}+n\frac{\partial\omega}{\partial n}=\omega \quad \ldots\ldots\ldots\ldots (104).$$

But ω is given by equation (103) which is of the form

$$f(l/\omega, m/\omega, n/\omega)=0,$$

and therefore writing f_1, f_2, f_3 for the differential coefficients of f with regard to l/ω, m/ω, n/ω, we have

$$(lf_1+mf_2+nf_3)\,d\omega=\omega(f_1dl+f_2dm+f_3dn),$$

so that equation (104) is satisfied.

Again, if we seek the point of contact of the plane

$$l(x-\alpha_0)+m(y-\beta_0)+n(z-\gamma_0)=\omega t,$$

with its envelope, when l, m, n vary and ω is a given function of l, m, n, we shall get, taking account of (104),

$$\left.\begin{aligned} x-\alpha_0-t\frac{\partial\omega}{\partial l}&=0\\ y-\beta_0-t\frac{\partial\omega}{\partial m}&=0\\ z-\gamma_0-t\frac{\partial\omega}{\partial n}&=0 \end{aligned}\right\} \quad \text{.....(105),}$$

and therefore the point

$$\left(\alpha_0+t\frac{\partial\omega}{\partial l},\quad \beta_0+t\frac{\partial\omega}{\partial m},\quad \gamma_0+t\frac{\partial\omega}{\partial n}\right)$$

is the point of contact at time t of the tangent plane parallel to the tangent plane initially at $(\alpha_0, \beta_0, \gamma_0)$.

The equations (105) are the equations of a straight line passing through $(\alpha_0, \beta_0, \gamma_0)$. This line is called the *ray* through $(\alpha_0, \beta_0, \gamma_0)$.

77. Wave-Surface.

We have shewn how the surface of discontinuity S at time t is connected with the initial surface S_0, viz. our equations shew that from every point P of S_0 we have to draw in a given direction, depending on that of the normal to S_0 at P, the ray through that point, and take on it a length proportional to the time and to a certain function of the direction of the normal to S_0 at P. This gives a construction for the points on S. Also the tangent plane to S at any such point is parallel to the tangent plane to S_0 at the corresponding point. This gives a construction for the tangent planes to S.

Now suppose the initial surface S_0 to be a small closed surface surrounding the point $(\alpha_0, \beta_0, \gamma_0)$. Then we have to draw normals in every direction from this surface and mark upon them lengths ωt where ω is a function of the direction of the normal given by equation (103). The planes drawn perpendicular to these normals at the points so found will envelope a surface, which Herr Christoffel calls the "central-surface" of the point $(\alpha_0, \beta_0, \gamma_0)$. That particular central-surface for which $t=1$ he calls the "wave-surface". If the wave-surface be constructed all other central-

surfaces are obtained from it by producing the radii vectores in the ratio $t : 1$.

78. Wave-Motion.

We can now give a sketch of the propagation of the disturbance through the medium. For this purpose we shall suppose that initially the part of the medium outside a certain surface S_0 is unstrained, and the medium within the surface is strained in a given manner. If then we draw the central-surface corresponding to time t for every point within S_0, these surfaces will have an envelope S, which will consist in general of six sheets, two for each value of ω. Fixing our attention on one value of ω and the corresponding sheets of S the motion of this type will be called a wave. Three such waves are propagated. The parts of the medium, not included between the two sheets of S, corresponding to a wave are at rest and unstrained. Every element of the medium when the wave reaches it takes suddenly the small displacement-velocity propagated with the wave. After a time depending on its position with respect to the original region of disturbance (the space within S_0), the wave will have passed over this element, and as the inner sheet of S passes over it the element suddenly loses the small velocity that it had, and returns to a position of rest and a configuration of no strain. The same thing happens for each of the three waves. The element, if it be far enough from S_0, is jerked into motion from rest, and returns impulsively to rest from motion by the action of three separate impulses, and its motion in each case lasts for a finite time depending on the size of S_0. In every case the whole motion depends simply on the form of the wave-surface and on the initial state.

The particular case of an isotropic solid is an example of a case in which the determinantal equation for ω has two equal roots, the k of our previous work. The reader will find it an instructive exercise to work out this case, and also the case of a medium whose energy-function is of the form

$$\tfrac{1}{2}\{A(e+f+g)^2+L(a^2-4fg)+M(b^2-4ge)+N(c^2-4ef)\},$$

which leads to Fresnel's wave-surface[1] and a sphere as the general wave-surface.

[1] See *Math. Papers of the late George Green*, pp. 303—305.

ON THE GENERAL THEORY OF THE FREE VIBRATIONS OF SOLIDS[1].

79. Determination of Principal Modes of Vibration.

Suppose a finite solid mass, bounded by a closed surface, and under the action of no bodily forces, is slightly disturbed, so that initially there is a given distribution of strain, displacement, and velocity, and suppose that the forces applied to the boundary are of the nature of constraints which do no work, as, for example, when a point of the surface is held fixed, or is constrained to move on a smooth fixed guiding curve or surface; the problem of determining the subsequent motion is a particular case of the general problem of determining the free vibrations of a system about a configuration of stable equilibrium.

We know that for such a system there are definite periods and types of vibration, and the type is determined by stating the ratio of the various displacements of all the points to the displacement of one of them in some particular direction. The displacements in any direction are in general continuous functions of position, and the amplitudes of the displacements in different directions are in a certain ratio. The whole motion is analysed into the sum of certain series of coexistent small motions which can be executed independently of one another. The motions of these types are called *principal modes of vibration.*

Now let u, v, w be the displacements, and suppose the solid is vibrating in a principal mode with a period $2\pi/p$. Then $p/2\pi$ is called the *frequency* and p the *speed* of the vibration. The functions u, v, w are for this mode proportional to simple harmonic functions of the time, *i.e.* of the form $\cos(pt + \epsilon)$.

Let p_r be any one of the speeds of principal modes and write ϕ_r for $\cos(p_r t + \epsilon_r)$, then we have to take

$$\left.\begin{aligned} u &= u_1\phi_1 + u_2\phi_2 + \ldots + u_r\phi_r + \ldots \\ v &= v_1\phi_1 + v_2\phi_2 + \ldots + v_r\phi_r + \ldots \\ w &= w_1\phi_1 + w_2\phi_2 + \ldots + w_r\phi_r + \ldots \end{aligned}\right\} \quad \ldots\ldots\ldots(106),$$

and the whole motion of the rth type is determined when u_r, v_r, w_r, p_r are known.

[1] Clebsch, *Theorie der Elasticität fester Korper*, and Lord Rayleigh's *Theory of Sound*, vol. I.

The quantities ϕ_r are called *normal coordinates,* and u_r, v_r, w_r *normal functions.* The general variational equation of motion is

$$\iiint \delta W dxdydz + \iiint \left\{ \rho \frac{\partial^2 u}{\partial t^2} \delta u + \rho \frac{\partial^2 v}{\partial t^2} \delta v + \rho \frac{\partial^2 w}{\partial t^2} \delta w \right\} dxdydz = 0 \quad \text{.....(107).}$$

If after performing the variations we put $u = u_r \phi_r$, $v = v_r \phi_r$, $w = w_r \phi_r$, and observe that

$$\frac{\partial^2 u}{\partial t^2} = u_r \phi_r p_r^2,$$

and so on, we see that ϕ_r will be a factor which can be removed from the resulting equations, and the part that arises from δW will be the same as if we substitute (u_r, v_r, w_r) for (u, v, w) in the expressions

$$\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z}, \quad \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z}, \quad \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z}.$$

If P_r, $Q_r \ldots U_r$ denote the values of P, $Q, \ldots U$ when u_r, v_r, w_r are substituted for u, v, w the equations of vibration are

$$\frac{\partial P_r}{\partial x} + \frac{\partial U_r}{\partial y} + \frac{\partial T_r}{\partial z} + \rho p_r^2 u_r = 0 \quad \text{.....(108),}$$

and two similar equations.

These are three partial differential equations of the second order for the determination of u_r, v_r, w_r.

In addition to these we have three boundary-conditions at every bounding surface. By substituting therein the values of u, v, w *i.e.* of u_r, v_r, w_r, since ϕ_r is a factor which may be removed, we shall obtain in general sufficient equations to determine the ratios of the unknown constants that occur in the solution, and one other equation generally transcendental which involves p_r. The values of p_r, that satisfy this equation, are the speeds of the possible principal oscillations of the system. The equation is generally referred to as the *frequency-equation.*

80. General Theorems on Vibrating Systems.

We can now use the general equation of vibration (107) to prove two theorems.

Theorem 1°. Suppose p_r and p_s are two roots of the frequency-equation and that the corresponding types are given by

$$\left.\begin{aligned} u &= \phi_r u_r, & v &= \phi_r v_r, & w &= \phi_r w_r \\ u &= \phi_s u_s, & v &= \phi_s v_s, & w &= \phi_s w_s \end{aligned}\right\} \ldots\ldots\ldots\ldots (109).$$

Then, in the variational equation, we may take u, v, w to be proportional to u_r, v_r, w_r, and δu, δv, δw to be proportional to u_s, v_s, w_s. Then remembering that

$$\frac{\partial^2 u}{\partial t^2} = u_r \frac{\partial^2 \phi_r}{\partial t^2} = -\, p_r^2 \phi_r u_r,$$

we have, omitting the time-factors,

$$p_r^2 \iiint \rho\,(u_r u_s + v_r v_s + w_r w_s)\, dxdydz = \iiint \delta W dxdydz.$$

Now
$$\iiint \delta W dxdydz = \iiint (P\delta e + \ldots)\, dxdydz$$
$$= \iiint (P_r e_s + Q_r f_s + \ldots + U_r c_s)\, dxdydz$$
$$= \iiint (P_s e_r + Q_s f_r + \ldots + U_s c_r)\, dxdydz,$$

by a general property of quadratic functions.

Thus $\iiint \delta W dxdydz$ is a symmetrical function of

$$(u_r,\ v_r,\ w_r),\quad (u_s,\ v_s,\ w_s),$$

and thus we shall obtain the same expression for this integral when we identify u, v, w with u_s, v_s, w_s and δu, δv, δw with u_r, v_r, w_r.

But proceeding as before we find that in this case

$$p_s^2 \iiint \rho\,(u_s u_r + v_s v_r + w_s w_r)\, dxdydz = \iiint \delta W dxdydz.$$

Hence

$$(p_r^2 - p_s^2) \iiint \rho\,(u_r u_s + v_r v_s + w_r w_s)\, dxdydz = 0,$$

and since $p_r^2 - p_s^2$ is not $= 0$ it follows that

$$\iiint \rho\,(u_r u_s + v_r v_s + w_r w_s)\, dxdydz = 0 \ldots\ldots\ldots\ldots (110).$$

This theorem enables us to determine the subsequent state in terms of the initial conditions by the method of Lord Rayleigh's *Theory of Sound*, art. 101.

Theorem 2°. We can shew that the frequency-equation for p^2 has always real positive roots.

For suppose if possible that $p_r^2 = \alpha + \iota\beta$ where α and β are real.

Then the equation will have a root $p_s^2 = \alpha - \iota\beta$.

We shall obtain two corresponding sets of normal functions, u_r and u_s..., which are conjugate imaginaries.

Thus $u_r u_s$ is the sum of two positive squares, and the same is true of $v_r v_s$ and $w_r w_s$, and therefore

$$\iiint \rho\,(u_r u_s + v_r v_s + w_r w_s)\,dxdydz,$$

is a sum of terms, which are all positive, and consequently this integral cannot vanish. The values of p^2 are therefore all real.

To shew that the roots are positive consider the integral

$$\iiint (u_r^2 + v_r^2 + w_r^2)\,dxdydz \quad \ldots\ldots\ldots\ldots\ldots\ldots (111),$$

which is always positive; this by (108) is equal to

$$-\rho^{-1}p_r^{-2}\iiint\left\{u_r\left(\frac{\partial P_r}{\partial x}+\frac{\partial U_r}{\partial y}+\frac{\partial T_r}{\partial z}\right)+\ldots+\ldots\right\}dxdydz,$$

and by Green's transformation this is

$$-\rho^{-1}p_r^{-2}\iint\{u_r(lP_r+mU_r+nT_r)+\ldots+\ldots\}\,dS$$
$$+\rho^{-1}p_r^{-2}\iiint\{P_re_r+Q_rf_r+R_rg_r+S_ra_r+T_rb_r+U_rc_r\}\,dxdydz.$$

Of this expression the first line vanishes identically[1] in virtue of the boundary conditions, and the second line is

$$2\rho^{-1}p_r^{-2}\iiint W_r\,dxdydz,$$

where W_r is the potential energy of strain per unit volume when the solid is vibrating in the rth normal mode. Hence the integral (111) is the product of p_r^{-2} and a quantity which is always positive, and therefore p_r^{-2} is also positive.

81. Load suddenly applied or suddenly reversed.

The theory of the vibrations of solids may be used to prove two theorems of great importance for the strength of materials. The first of these is that the strain produced by a load suddenly applied may be twice as great as that produced by the gradual application of the same load; the second is that, if the load be suddenly reversed, the strain may be trebled.

To prove the first theorem, we observe that, if a load be suddenly applied to an elastic system, the system will be thrown into a state of vibration about a certain equilibrium configuration, viz. that which the system would take if the load were applied

[1] If the surfaces be not free there are additional surface-terms for the work done by the surface-tractions, and the surface-integral again vanishes.

gradually. The initial state is one in which the energy is purely potential, and, as there is no elastic stress, this energy is due simply to the position of the elastic solid in the field of force constituting the load. If the initial position be a possible position of instantaneous rest in a normal mode of oscillation of the system, then the system will oscillate in that normal mode, and the configuration at the end of a quarter of a period will be the equilibrium configuration, *i.e.* the displacement from the equilibrium configuration will be zero; at the end of a half-period, it will be equal and opposite to that in the initial position. The maximum displacement from the initial configuration will therefore be twice that in the equilibrium configuration. If the system, when left to itself under the suddenly applied load, do not oscillate in a normal mode the strain will be less than twice that in the equilibrium configuration, since the system never passes into a configuration in which the energy is purely potential.

The proof of the second theorem is similar. The system being held strained in a configuration of equilibrium, the load is suddenly reversed, and the new position of equilibrium is one in which all the displacements are reversed. This is the position about which the system oscillates. If it oscillate in a normal mode the maximum displacement from the equilibrium configuration is double the initial displacement from the configuration of no strain; and, at the instant when the displacement from the equilibrium configuration is a maximum, the displacement from the configuration of no strain is three times that which would obtain in the equilibrium configuration.

A typical example of the first theorem is the case of an elastic string, to which a weight is suddenly attached. The greatest extension of the string is double that which it has, when statically supporting the weight.

A typical example of the second theorem is the case of a cylindrical shaft held twisted. If the twisting couple be suddenly reversed the greatest shear can be three times that which originally accompanied the twist.

CHAPTER VI.

THE EQUILIBRIUM OF BEAMS. SAINT-VENANT'S PROBLEM[1].

82. The Semi-inverse Method.

It seems in the first place appropriate to explain the semi-inverse method of solution adopted by Saint-Venant, and to give the reasons which led to its adoption, and this leads us to speak of the theory of beams in practical use by engineers and others before the publication of his researches on the subject. Let us for example consider flexure. The problem of determining the resistance of a beam to flexure, when one end is built into a wall, while the other end supports a weight, is the oldest problem of the

[1] The following among other authorities may be consulted.

Saint-Venant. 'Mémoire sur la Torsion des Prismes, avec des considérations sur leur flexion, ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément'. *Mém. des Savants étrangers*, 1855. Also 'Mémoire sur la flexion des prismes, sur les glissements...qui l'accompagnent..., et sur la forme courbe affectée...par leurs sections transversales primitivement planes', Liouville's *Journal*, 1856. Also 'Sur une formule donnant approximativement le moment de torsion', *Comptes Rendus*, LXXXVIII. 1879.

Clebsch. *Theorie der Elasticität fester Körper.*

Thomson and Tait. *Natural Philosophy*, vol. I. part II.

Basset. *Hydrodynamics*, vol. I.

Pearson. 'On the Flexure of Heavy Beams subjected to Continuous Systems of Load'. *Quarterly Journal*, 1890.

Navier. *Léçons sur l'application de la Mécanique...*, 3rd edition, 1863, with notes by Saint-Venant.

Voigt. 'Allgemeine Formeln für die Bestimmung der Elasticitätsconstanten von Krystallen...', Wiedemann's *Annalen*, XVI. 1882, and 'Ueber die Torsion eines rechteckigen Prismas aus homogener krystallinischer Substanz', Wiedemann's *Annalen*, XXIX. 1886.

subject of Elasticity, and had received discussion even before the discovery of Hooke's Law. This problem continued to attract the attention of mathematicians, and was the subject of researches by Coulomb, Euler, the Bernoullis, Navier, and many others, but for practical purposes most simple questions of flexure may be regarded as settled by Saint-Venant's solution. The method of the older mathematicians was to suppose the resistance to flexure to be the resultant of the stresses arising from the extensions and contractions which the fibres of the bent beam undergo, taking no account of the shears of the cross-sections, or the distortion of these sections, by which the bending is generally accompanied. Saint-Venant pointed out that in general the method is inadequate, as its hypotheses are false and some of its conclusions erroneous, but he set himself to discover whether in this and similar cases a solution of the equations of elasticity could be obtained, which, leaving intact some of these hypotheses and conclusions, should yet be applicable to a large majority of practical problems. The semi-inverse method of solution consists in imposing a restriction on the generality of the stress within the solid in accordance with a result based on some theory not derived from a solution of the general equations. In the particular case of beams, the conclusion borrowed from the older theories is that each fibre of the beam parallel to the generators of its bounding surface, is deformed by forces acting on its ends alone, and suffers no traction from neighbouring fibres.

We are to suppose, then, a beam of cylindrical form with plane ends perpendicular to its axis, to be subject to the action of forces on its plane ends, while no traction is exerted on its cylindrical bounding surface, and we are further to suppose that there is no stress across any plane parallel to the axis. To make our work as generally applicable as possible we shall assume that the material of the beam has three rectangular planes of symmetry[1],

[1] Saint-Venant began with a solid which has one plane of symmetry only, perpendicular to the axis of the beam, but introduced the other two planes afterwards to simplify the work. The student reading the subject for the first time is advised to work over all the general theory for the case of an isotropic beam.

As a further example of the analysis in the next article it may be shewn that, if Saint-Venant's stress conditions be imposed and the beam be supposed vibrating, equation (12) will be satisfied, and equations (11) and (13) become

$$\frac{\partial^3 w}{\partial z^3} = \frac{\rho(1+2\sigma)}{2\mu(1+\sigma)} \frac{\partial^3 w}{\partial z\, \partial t^2} \qquad \text{.....(11)},$$

two of which intersect in the axis of the beam, but we shall not at first take it to be isotropic.

83. Equations of the Problem.

Take then the axis of z parallel to the length of the beam, and suppose that it is the line of centres of inertia of the normal sections in the unstrained state, and suppose the energy-function of the material when strained to be W, where

$$2W = (A, B, C, F, G, H)(e, f, g)^2 + La^2 + Mb^2 + Nc^2 \ldots (1),$$

so that the stresses are given by the equations

$$\left.\begin{aligned} P &= Ae + Hf + Gg, & S &= La \\ Q &= He + Bf + Fg, & T &= Mb \\ R &= Ge + Ff + Cg, & U &= Nc \end{aligned}\right\} \ldots\ldots\ldots\ldots (2).$$

The stress-conditions imposed by the semi-inverse method are

$$P = Q = U = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

and

$$\frac{\partial^3 w}{\partial z \partial x^2} = -\frac{\sigma\rho}{2\mu(1+\sigma)} \frac{\partial^3 w}{\partial z \partial t^2} = \frac{\partial^3 w}{\partial z \partial y^2} \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

where σ is the Poisson's ratio of the material, supposed isotropic, and μ is the rigidity. Equations (11), (12), and (13) cannot be satisfied unless $\frac{\partial w}{\partial z}$ is independent of x and y. The equation corresponding to (10) is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2 w}{\partial z^2} = \frac{\rho}{\mu} \frac{\partial^2 w}{\partial t^2} \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

and, on differentiating this with respect to z, we find incompatible equations for $\frac{\partial w}{\partial z}$ as a function of z and t. Hence $\frac{\partial w}{\partial z}$ must be zero; and then, since (7) holds, u is a function of z, y, t and v of z, x, t; and using (8) and (6) and the equations corresponding to (9), we shall find that

$$u = -\tau y e^{in(z-bt)},$$

$$v = \tau x e^{in(z-bt)},$$

where τ is a constant, and $b^2 = \mu/\rho$, and the boundary condition can be satisfied only when the boundary is a circle.

Thus a circular cylinder can execute purely torsional vibrations under Saint-Venant's stress condition; and, with this exception, the only vibrations under this condition are given by

$$u = 0, \quad v = 0, \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\rho}{\mu} \frac{\partial^2 w}{\partial t^2},$$

and the boundary condition

$$l\frac{\partial w}{\partial x} + m\frac{\partial w}{\partial y} = 0.$$

These are similar to the vibrations of water in a cylindrical tank whose curved surface coincides with that of the beam.

We have therefore for the equations of equilibrium

$$\left.\begin{aligned} \frac{\partial T}{\partial z} = 0, \quad & \frac{\partial S}{\partial z} = 0 \\ \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} = 0 \quad & \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and the only condition at the cylindrical boundary, which is not satisfied identically, is

$$lT + mS = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where l, m are the cosines of the angles which the normal to the boundary drawn outwards makes with the axes of x and y.

We may also suppose the geometrical conditions satisfied at the origin to be

$$u = 0, \quad v = 0, \quad w = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad \frac{\partial u}{\partial y} = 0 \ldots (6).$$

Then the problem consists in the discovery of the most general solution of the equations (3), (4), (5), (6), and the determination of the consequent amount and distribution of force over the plane ends of the cylinder[2].

84. Equations for the displacements.

Since $P = 0$, $Q = 0$, we must have

$$\left.\begin{aligned} \frac{\partial u}{\partial x} = -\frac{BG - HF}{AB - H^2} \frac{\partial w}{\partial z} = -\sigma_1 \frac{\partial w}{\partial z}, \text{ say} \\ \frac{\partial v}{\partial y} = -\frac{AF - GH}{AB - H^2} \frac{\partial w}{\partial z} = -\sigma_2 \frac{\partial w}{\partial z}, \text{ say} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7),$$

where σ_1 and σ_2 are the ratios of lateral contraction, parallel to x and y respectively, to longitudinal extension parallel to z.

Also, since $U = 0$, we must have

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

[1] These equations denote that the origin is supposed held fixed, that the element of the axis of the beam at the origin retains its primitive direction, and the element of the plane through it and the axis y retains its primitive direction. If any other conditions be imposed at the origin the displacements consist of those that we shall obtain combined with a suitable rigid body displacement.

[2] The problem in this form was first considered by Clebsch. (See Introduction.)

And, since $\dfrac{\partial T}{\partial z}=0$ and $\dfrac{\partial S}{\partial z}=0$, we must have

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x \partial z}&=0\\ \frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial y \partial z}&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

The third equation of (4) is

$$M\left(\frac{\partial^2 u}{\partial z \partial x}+\frac{\partial^2 w}{\partial x^2}\right)+L\left(\frac{\partial^2 v}{\partial y \partial z}+\frac{\partial^2 w}{\partial y^2}\right)+G\frac{\partial^2 u}{\partial z \partial x}+F\frac{\partial^2 v}{\partial z \partial y}+C\frac{\partial^2 w}{\partial z^2}=0,$$

or by (7),

$$M\frac{\partial^2 w}{\partial x^2}+L\frac{\partial^2 w}{\partial y^2}+[C-(M+G)\sigma_1-(L+F)\sigma_2]\frac{\partial^2 w}{\partial z^2}=0\ldots(10).$$

85. Determination of the form of u and v.

Differentiate (10) with respect to z, the equations of (9) with respect to x and y respectively, and use (7) to eliminate u and v, and we get

$$\frac{\partial^3 w}{\partial z^3}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Differentiate equations (9) with respect to y and x, add, and use (8), and we get

$$\frac{\partial^3 w}{\partial x \partial y \partial z}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Differentiate equations (9) with respect to x and y, then, using (7) and (11), we get

$$\frac{\partial^3 w}{\partial x^2 \partial z}=0,\quad \frac{\partial^3 w}{\partial y^2 \partial z}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

It thus appears that $\dfrac{\partial w}{\partial z}$ is linear in z, and linear in x and y separately, and therefore

$$\frac{\partial w}{\partial z}=(\alpha+\alpha_1 x+\alpha_2 y)+z(\beta+\beta_1 x+\beta_2 y)\ldots\ldots\ldots(14),$$

where the α's and β's are constants; and the only possible forms for u and v that satisfy (7), (9) and (14) are

$$u=-\sigma_1(\alpha x+\tfrac{1}{2}\alpha_1 x^2+\alpha_2 xy)-\sigma_1 z(\beta x+\tfrac{1}{2}\beta_1 x^2+\beta_2 xy)\\ -\tfrac{1}{2}\alpha_1 z^2-\tfrac{1}{6}\beta_1 z^3+u_0+u_1 z,$$

$$v=-\sigma_2(\alpha y+\alpha_1 xy+\tfrac{1}{2}\alpha_2 y^2)-\sigma_2 z(\beta y+\beta_1 xy+\tfrac{1}{2}\beta_2 y^2)\\ -\tfrac{1}{2}\alpha_2 z^2-\tfrac{1}{6}\beta_2 z^3+v_0+v_1 z,$$

where u_0 and u_1 are functions of y, and v_0 and v_1 functions of x.

Now (8) shews that the equation

$$-\sigma_1\alpha_2 x - \sigma_1\beta_2 zx + \frac{\partial u_0}{\partial y} + z\frac{\partial u_1}{\partial y} + \left[-\sigma_2\alpha_1 y - \sigma_2\beta_1 zy + \frac{\partial v_0}{\partial x} + z\frac{\partial v_1}{\partial x}\right] = 0$$

is identically satisfied whatever z may be, and therefore

$$\begin{aligned} u_0 &= \alpha' + \alpha_0 y + \tfrac{1}{2}\sigma_2\alpha_1 y^2, \\ u_1 &= \beta' + \beta_0 y + \tfrac{1}{2}\sigma_2\beta_1 y^2, \\ v_0 &= \alpha'' - \alpha_0 x + \tfrac{1}{2}\sigma_1\alpha_2 x^2, \\ v_1 &= \beta'' - \beta_0 x + \tfrac{1}{2}\sigma_1\beta_2 x^2, \end{aligned}$$

where all the α's and β's are constants.

Hence, using equation (8) and the conditions (6) at the origin, we find for u and v the forms

$$\left.\begin{aligned} u = &-\sigma_1(\alpha x + \tfrac{1}{2}\alpha_1 x^2 + \alpha_2 xy) - \sigma_1 z(\beta x + \tfrac{1}{2}\beta_1 x^2 + \beta_2 xy) \\ &- \tfrac{1}{2}\alpha_1 z^2 - \tfrac{1}{6}\beta_1 z^3 + \tfrac{1}{2}\sigma_2\alpha_1 y^2 + z(\beta_0 y + \tfrac{1}{2}\sigma_2\beta_1 y^2), \\ v = &-\sigma_2(\alpha y + \alpha_1 xy + \tfrac{1}{2}\alpha_2 y^2) - \sigma_2 z(\beta y + \beta_1 xy + \tfrac{1}{2}\beta_2 y^2) \\ &- \tfrac{1}{2}\alpha_2 z^2 - \tfrac{1}{6}\beta_2 z^3 + \tfrac{1}{2}\sigma_1\alpha_2 x^2 - z(\beta_0 x - \tfrac{1}{2}\sigma_1\beta_2 x^2) \end{aligned}\right\} \ldots(15).$$

86. Determination of w.

Returning now to equation (14) we find

$$w = z(\alpha + \alpha_1 x + \alpha_2 y) + \tfrac{1}{2}z^2(\beta + \beta_1 x + \beta_2 y) + w' \ldots\ldots(16),$$

where w' is a function of x and y which satisfies the equation

$$M\frac{\partial^2 w'}{\partial x^2} + L\frac{\partial^2 w'}{\partial y^2} + [C - (M+G)\sigma_1 - (L+F)\sigma_2](\beta + \beta_1 x + \beta_2 y) = 0.$$

A particular solution is

$$w' = -[C - (M+G)\sigma_1 - (L+F)\sigma_2]\left[\tfrac{1}{2}\beta\left(\frac{x^2+y^2}{M+L}\right) + \tfrac{1}{2}\beta_1\frac{xy^2}{L} + \tfrac{1}{2}\beta_2\frac{yx^2}{M}\right] \ldots(17),$$

and therefore w consists of (16), in which w' is given by (17), and a solution as general as may be required of the equation

$$M\frac{\partial^2\phi}{\partial x^2} + L\frac{\partial^2\phi}{\partial y^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(18).$$

Thus, if ϕ be the solution of this equation required to satisfy the boundary-conditions, w is given by the equation

$$w = z(\alpha + \alpha_1 x + \alpha_2 y) + \tfrac{1}{2}z^2(\beta + \beta_1 x + \beta_2 y) + \phi - K\left\{\tfrac{1}{2}\beta\left(\frac{x^2+y^2}{M+L}\right) + \tfrac{1}{2}\beta_1\frac{xy^2}{L} + \tfrac{1}{2}\beta_2\frac{yx^2}{M}\right\} \ldots(19),$$

where

$$K = C - (M+G)\sigma_1 - (L+F)\sigma_2 \ldots\ldots\ldots\ldots(20).$$

87. Boundary-condition.

The stresses corresponding to these displacements are by (2)

$$\left.\begin{aligned} S = L\left[\frac{\partial\phi}{\partial y} - \beta_0 x - \beta y\left(\sigma_2 + \frac{K}{L+M}\right) - \beta_1 xy\left(\sigma_2 + \frac{K}{L}\right) - \tfrac{1}{2}\beta_2\sigma_2 y^2\right. \\ \left. + \tfrac{1}{2}\beta_2 x^2\left(\sigma_1 - \frac{K}{M}\right)\right] \\ T = M\left[\frac{\partial\phi}{\partial x} + \beta_0 y - \beta x\left(\sigma_1 + \frac{K}{L+M}\right) - \beta_2 xy\left(\sigma_1 + \frac{K}{M}\right) - \tfrac{1}{2}\beta_1\sigma_1 x^2\right. \\ \left. + \tfrac{1}{2}\beta_1 y^2\left(\sigma_2 - \frac{K}{L}\right)\right] \\ R = (C - G\sigma_1 - F\sigma_2)\left[\alpha + \alpha_1 x + \alpha_2 y + z(\beta + \beta_1 x + \beta_2 y)\right] \end{aligned}\right\} \quad (21).$$

The condition (5) at the cylindrical boundary becomes

$$\begin{aligned} \left(Ml\frac{\partial\phi}{\partial x} + Lm\frac{\partial\phi}{\partial y}\right) &= \beta_0(Lmx - Mly) \\ &+ \beta\left[Lmy\left(\sigma_2 + \frac{K}{L+M}\right) + Mlx\left(\sigma_1 + \frac{K}{L+M}\right)\right] \\ &+ \beta_1\left[Ml\tfrac{1}{2}\sigma_1 x^2 + Lmxy\left(\sigma_2 + \frac{K}{L}\right) - Ml\tfrac{1}{2}y^2\left(\sigma_2 - \frac{K}{L}\right)\right] \\ &+ \beta_2\left[Lm\tfrac{1}{2}\sigma_2 y^2 + Mlxy\left(\sigma_1 + \frac{K}{M}\right) - Lm\tfrac{1}{2}x^2\left(\sigma_1 - \frac{K}{M}\right)\right] \quad \ldots(22). \end{aligned}$$

This condition holds at every point of the boundary. If therefore we multiply by the element of arc of the boundary ds, and integrate round the boundary; and transform the line-integrals into surface-integrals over the normal section, the left-hand side vanishes identically by equation (18), and the right-hand side becomes

$$\begin{aligned} &\beta\iint\left[L\left(\sigma_2 + \frac{K}{L+M}\right) + M\left(\sigma_1 + \frac{K}{L+M}\right)\right]dxdy \\ &+ \beta_1\iint\left[M\sigma_1 + L\left(\sigma_2 + \frac{K}{L}\right)\right]xdxdy \\ &+ \beta_2\iint\left[L\sigma_2 + M\left(\sigma_1 + \frac{K}{M}\right)\right]ydxdy. \end{aligned}$$

The last two terms vanish if we take the axis z to pass through the centres of inertia of the normal sections, and it follows that in general β must be zero, the term in β being

$$\beta(C - G\sigma_1 - F\sigma_2)\,[\text{area of normal section}].$$

The coefficient $C - G\sigma_1 - F\sigma_2$ can be shewn to be the Young's modulus E of the material for traction parallel to z (ch. III. arts. 43 and 44).

88. Simplified forms of the Displacements and Stresses.

As we have shewn that $\beta = 0$ we may rewrite our equations as follows:

$$\left.\begin{aligned} u &= -\sigma_1 \alpha x - \tfrac{1}{2}\alpha_1(\sigma_1 x^2 - \sigma_2 y^2) - \sigma_1 \alpha_2 xy + \beta_0 yz \\ &\quad - \tfrac{1}{2}\beta_1 z(\sigma_1 x^2 - \sigma_2 y^2) - \sigma_1 \beta_2 xyz - \tfrac{1}{2}\alpha_1 z^2 - \tfrac{1}{6}\beta_1 z^3 \\ v &= -\sigma_2 \alpha y - \sigma_2 \alpha_1 xy + \tfrac{1}{2}\alpha_2(\sigma_1 x^2 - \sigma_2 y^2) - \beta_0 zx \\ &\quad - \sigma_2 \beta_1 xyz + \tfrac{1}{2}\beta_2 z(\sigma_1 x^2 - \sigma_2 y^2) - \tfrac{1}{2}\alpha_2 z^2 - \tfrac{1}{6}\beta_2 z^3 \\ w &= z(\alpha + \alpha_1 x + \alpha_2 y) + \tfrac{1}{2}z^2(\beta_1 x + \beta_2 y) + \phi \\ &\quad - \tfrac{1}{2}(E - M\sigma_1 - L\sigma_2)(\beta_1 xy^2/L + \beta_2 yx^2/M) \end{aligned}\right\} \ldots\ldots(23),$$

where ϕ satisfies the equation (18), viz.:

$$M\frac{\partial^2\phi}{\partial x^2} + L\frac{\partial^2\phi}{\partial y^2} = 0$$

at all points of any normal section, and the condition

$$\begin{aligned} Ml\frac{\partial\phi}{\partial x} + Lm\frac{\partial\phi}{\partial y} &= \beta_0(Lmx - Mly) \\ &\quad + \beta_1\left[\tfrac{1}{2}Ml\sigma_1 x^2 + \frac{E - M\sigma_1}{L}(Lmxy + \tfrac{1}{2}Mly^2) - Ml\sigma_2 y^2\right] \\ &\quad + \beta_2\left[\tfrac{1}{2}Lm\sigma_2 y^2 + \frac{E - L\sigma_2}{M}(Mlxy + \tfrac{1}{2}Lmx^2) - Lm\sigma_1 x^2\right] \ldots(24) \end{aligned}$$

at all points of the cylindrical boundary. The stresses at any point are

$$\left.\begin{aligned} S &= L\left[\frac{\partial\phi}{\partial y} - \beta_0 x - \frac{E - M\sigma_1}{L}\beta_1 xy \right. \\ &\qquad \left. - \tfrac{1}{2}\beta_2\left(\sigma_2 y^2 + \frac{E - L\sigma_2}{M}x^2 - 2\sigma_1 x^2\right)\right] \\ T &= M\left[\frac{\partial\phi}{\partial x} + \beta_0 y - \tfrac{1}{2}\beta_1\left(\sigma_1 x^2 + \frac{E - M\sigma_1}{L}y^2 - 2\sigma_2 y^2\right) \right. \\ &\qquad \left. - \frac{E - L\sigma_2}{M}\beta_2 xy\right] \\ R &= E[\alpha + \alpha_1 x + \alpha_2 y + z(\beta_1 x + \beta_2 y)] \end{aligned}\right\} \ldots\ldots(25).$$

89. Character of the solution.

The solution is linear in 6 independent arbitrary constants, α, α_1, α_2, β_0, β_1, β_2 and we may suppose all but one of these to vanish. Thus we shall have a solution involving α, a solution

involving α_1, and so on. Of these the solution involving α_2 can be obtained from that involving α_1 by changing x into y and making appropriate changes in the constants σ_1, L,..., and the solution involving β_2 is similarly related to that involving β_1.

The solution involving α corresponds to simple *extension* of the beam.

The solution involving α_1 corresponds to *uniform flexure* in the plane of (x, z); for the equations of the line of particles initially coinciding with the axis of z become

$$y = 0, \quad x = -\tfrac{1}{2}\alpha_1 z^2.$$

As all the β's vanish, $\phi = 0$, and the resultant stress across any section reduces to a couple about the axis of y.

The solution involving β_0 corresponds to *torsion* about the z axis; for this axis retains its primitive position, and every normal section is rotated through an angle $-\beta_0 z$. The resultant stress across any section reduces to a couple about the axis z.

The solution involving β_1 corresponds to *non-uniform flexure* in the plane (x, z); for the equations of the line of particles initially coinciding with the axis of z become

$$y = 0, \quad x = -\tfrac{1}{6}\beta_1 z^3,$$

and the resultant stress is of a more general character than in the other solutions. We shall shew that, by a combination of this with the previous cases, it is possible to make the stress reduce to transverse force parallel to the axis of x.

90. Extension of the cylinder.

The displacements are

$$\left.\begin{aligned} u &= -\sigma_1 \alpha x \\ v &= -\sigma_2 \alpha y \\ w &= \alpha z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (26),$$

where α is the extension of the beam.

The only stress that is different from zero is R, and we have

$$R = E\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27),$$

where E is Young's modulus of the material for extension parallel to the axis of the beam.

The resultant stress across any normal section is

$$E\omega\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (28),$$

where ω is the area of the cross-section.

91. Uniform Flexure.

Suppose all the constants except α_1 to vanish. The displacements are

$$\left.\begin{aligned} u &= -\tfrac{1}{2}\alpha_1(z^2 + \sigma_1 x^2 - \sigma_2 y^2) \\ v &= -\alpha_1\sigma_2 xy \\ w &= \alpha_1 xz \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (29).$$

All the stresses except R vanish, and

$$R = E\alpha_1 x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

The resultant stress across any section reduces to a couple about the axis y of amount

$$-E\alpha_1 I_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31),$$

where I_1 is the moment of inertia of the cross-section with respect to the axis y.

The central-line is deformed into the curve

$$\left.\begin{aligned} x &= -\tfrac{1}{2}\alpha_1 z^2 \\ y &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

which is an arc of a parabola, but as the bending is very small we may regard it as an arc of a circle of large radius $\dfrac{1}{\alpha_1}$.

The change of position of the cross-sections is given by the equation

$$w = \alpha_1 xz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (33),$$

which shews that any section $z = z_0$ becomes

$$z = z_0(1 + \alpha_1 x) \ldots\ldots\ldots\ldots\ldots\ldots (34),$$

so that this section is turned round the axis y through an angle $z_0\alpha_1$ and remains plane.

The distortion of the shape of the cross-sections depends on u and v. Consider the beam rectangular, and let $2a$ and $2b$ be the lengths of the sides. Then, referred to the new position of the centre of inertia the sides

$$x = \pm a, \quad \text{and} \quad y = \pm b$$

become $$x \mp a = -\tfrac{1}{2}\alpha_1(\sigma_1 a^2 - \sigma_2 y^2),$$

and $$y \mp b = -\sigma_2\alpha_1 bx.$$

The latter are straight, and intersect $y \mp b = 0$ in $x = 0$, and are inclined to these at a small angle $\sigma_2 b\alpha_1$; the former are arcs of parabolas, and their curvature is $\sigma_2\alpha_1$.

If, to fix ideas, we draw the axis z horizontal, and perpendicular

to the paper, and the axis x in any section vertically upwards, the central line will be approximately part of a circle, of radius $\frac{1}{\alpha_1}$ whose centre is vertically above the origin (α_1 being taken negative). The shape of the cross-section[1] is given in the figure in which all lines parallel to AB or CD become approximately arcs of circles with their concavities downwards or in the opposite direction to that of the central line, and the upper and lower plane faces of the bar become anticlastic surfaces with their principal curvatures in the ratio $1 : \sigma_2$.

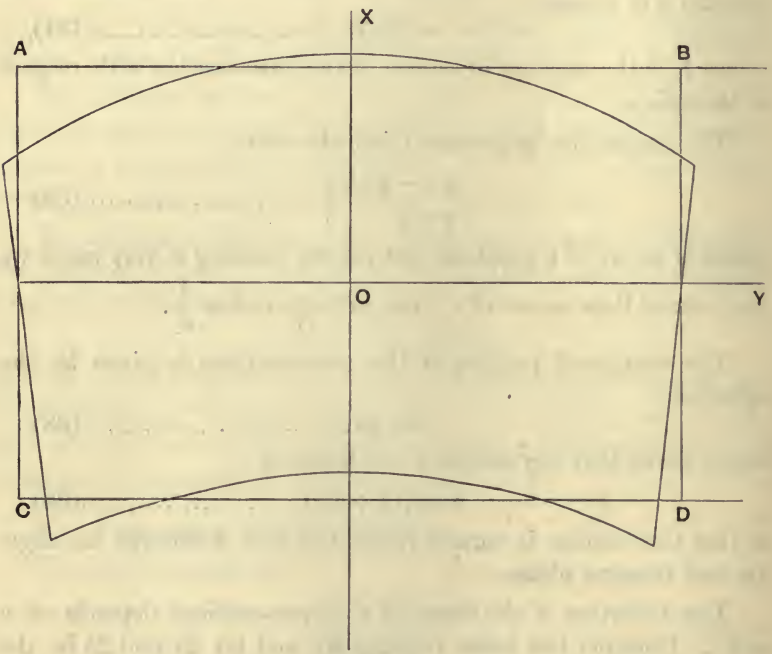

Fig. 10.

If this figure be supposed raised through the distance necessary to bring its centre on to the circle formed by the central-line, and turned round the axis y until the axis x produced passes through the centre of this circle, we shall have a complete representation of the changes of form and position involved, and the representation applies equally well to the distortion of a beam of rectangular section and to the distortion of a rectangular portion of a beam of any section.

[1] The curvature in the figure is much exaggerated.

92. Torsion of the Cylinder.

Let us write

$$\beta_0 = -\tau,$$

so that τz is the angle turned through by any section at a distance z from the fixed end. Then the displacements are

$$\left.\begin{aligned} u &= -\tau yz \\ v &= \tau zx \\ w &= \phi \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35),$$

where ϕ satisfies the differential equation

$$M\frac{\partial^2\phi}{\partial x^2} + L\frac{\partial^2\phi}{\partial y^2} = 0 \ldots\ldots\ldots\ldots\ldots (36),$$

and the boundary-condition

$$Ml\frac{\partial\phi}{\partial x} + Lm\frac{\partial\phi}{\partial y} = \tau(Mly - Lmx) \ldots\ldots\ldots (37).$$

The resultant stress across any section has components

$$M\iint\left(\frac{\partial\phi}{\partial x} - \tau y\right)dxdy \quad \text{parallel to } x,$$

$$L\iint\left(\frac{\partial\phi}{\partial y} + \tau x\right)dxdy \quad \text{parallel to } y.$$

The first of these may be written

$$\iint\left[M\frac{\partial}{\partial x}\left\{x\left(\frac{\partial\phi}{\partial x} - \tau y\right)\right\} + L\frac{\partial}{\partial y}\left\{x\left(\frac{\partial\phi}{\partial y} + \tau x\right)\right\}\right]dxdy,$$

since (36) holds at all points to which the integration extends; and this can be transformed to

$$\int x\left\{Ml\frac{\partial\phi}{\partial x} + Lm\frac{\partial\phi}{\partial y} - \tau(Mly - Lmx)\right\}ds,$$

where ds is an element of arc of the boundary. The line-integral vanishes identically, since ϕ satisfies the boundary-condition (37), and thus the resultant stress parallel to x vanishes. In like manner the resultant stress parallel to y vanishes. The stress therefore reduces to a couple

$$\iint\left\{Lx\left(\frac{\partial\phi}{\partial y} + \tau x\right) - My\left(\frac{\partial\phi}{\partial x} - \tau y\right)\right\}dxdy$$

$$= \tau(LI_1 + MI_2) + \iint\left(Lx\frac{\partial\phi}{\partial y} - My\frac{\partial\phi}{\partial x}\right)dxdy \ldots\ldots (38),$$

where I_1 and I_2 are the moments of inertia of the cross-section with respect to the axes of y and x.

93. Symmetrical Case.

If the two principal rigidities L and M be equal the theory is simplified. Taking $L = M = \mu$[1], we find that the stress gives rise to a couple about the axis z of amount

$$\mu \iint \left\{ x\left(\frac{\partial \phi}{\partial y} + \tau x\right) - y\left(\frac{\partial \phi}{\partial x} - \tau y\right) \right\} dxdy$$

or

$$\mu\tau I + \mu \iint \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial y} \right) dxdy \ldots\ldots\ldots (39),$$

where I is the moment of inertia of the section about the axis z.

If we suppose

$$\iint \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} \right) dxdy = (q-1)\tau \iint (x^2 + y^2)\, dxdy,$$

the couple will be $q\mu\tau I$. The quantity $q\mu I$ is called the *torsional rigidity* of the prism.

These results suggest two considerations. The first is a comparison with previous theories. The predecessors of Saint-Venant had generally supposed that, in every case of torsion, the stress at any section reduced to a couple about the axis of the cylinder, whose amount is $\mu\tau I$ where τ is the amount of the shear. In their work the distortion of the cross-section, implied by the existence of ϕ, was neglected. It is only for the circular cylinder that ϕ vanishes, and the property assumed is a unique property of the circular cylinder. Saint-Venant by introducing ϕ shewed that the couple is only proportional to that assumed by his predecessors, the coefficient q depending on the size and form of the section. This coefficient is now called Saint-Venant's "torsion-factor".

94. Hydrodynamical Analogy.

The second consideration is that there is an analogous problem in Hydrodynamics, viz.: it will appear that the solution can be derived from that for the motion of frictionless liquid in a rotating cylindrical vessel.

Let Φ be the velocity-potential of the liquid, ω the angular velocity of rotation, then the conditions to be satisfied are

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \ldots\ldots\ldots\ldots\ldots (40)$$

[1] We shall retain this supposition till the end of art. 102.

at all points of the section, and the boundary-condition

$$l\frac{\partial\Phi}{\partial x}+m\frac{\partial\Phi}{\partial y}=-l\omega y+m\omega x \quad \ldots\ldots\ldots\ldots\ldots (41)$$

at all points of the bounding curve.

So that $$\Phi:\phi=-\omega:\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42).$$

In the hydrodynamical problem the whole momentum of the liquid is angular and the moment of momentum is

$$\rho\iint\left(x\frac{\partial\Phi}{dy}-y\frac{\partial\Phi}{dx}\right)dxdy \quad \ldots\ldots\ldots\ldots\ldots (43),$$

where ρ is the density, and this is $-\omega\rho\,(q-1)\,I$.

If we suppose the vessel constructed of such material that its moment of inertia about the axis of the cylinder is $-\rho I$, the whole impulse required to start the motion will be $-\rho\omega qI$, so that, identifying ρ and μ, the impulse in the hydrodynamical problem will be identical with the couple in the elastic problem. The hydrodynamical problem is however no longer a real physical problem as it involves a negative distribution of matter on the surface of the cylinder.

The hydrodynamical analogy suggests the method to be followed in the solution of the torsion-problem. We know that in irrotational motion of a liquid in two dimensions there exists a stream-function Ψ, which is the conjugate-function of Φ with respect to x and y, and that the value of Ψ is given at the boundary, and it is in general simpler to solve the equation $\dfrac{\partial^2V}{\partial x^2}+\dfrac{\partial^2V}{\partial y^2}=0$ when the value of V is given at any boundary than when the value of $\dfrac{\partial V}{\partial n}$ (rate of variation in the direction of the normal) is given at the boundary.

We shall accordingly suppose that ϕ and ψ are conjugate functions of x and y, so that $\phi+\iota\psi$ is a function of the complex variable $x+\iota y$, then we know that ϕ and ψ satisfy the same partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\begin{matrix}\phi\\ \psi\end{matrix}=0,$$

and we have $$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y},\quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots (44).$$

We have to obtain the boundary-condition for ψ.

Let ds be an element of arc of the bounding curve of a normal section of the cylinder, measured in that direction in which the

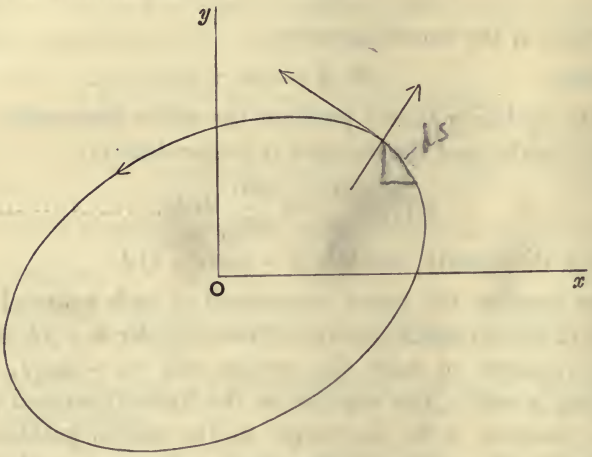

Fig. 11.

curve must be described in order that its area may be always to the left of the boundary, then

$$l = \frac{dy}{ds}, \quad m = -\frac{dx}{ds} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

are the cosines of the angles which the normal to the boundary makes with the axes, and the boundary-condition becomes

$$\frac{dy}{ds}\frac{\partial \psi}{\partial y} + \frac{dx}{ds}\frac{\partial \psi}{\partial x} = \tau\left(x\frac{dx}{ds} + y\frac{dy}{ds}\right).$$

We may obtain the integral of this equation in the form

$$\psi = \tfrac{1}{2}\tau(x^2 + y^2) + C \qquad \ldots\ldots\ldots\ldots\ldots\ldots (46),$$

where C is an arbitrary constant.

Thus the problem is reduced to finding a solution of the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

at all points of the section, subject to the condition

$$\psi = \tfrac{1}{2}\tau(x^2 + y^2) + C \qquad \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

at all points of the boundary.

95. Hydrodynamical Analogy, continued.

There is another form of the hydrodynamical analogy in which the problem of torsion is compared with that of a liquid circulating with uniform spin in a fixed cylindrical vessel coinciding with the surface of the twisted prism. In this form the moment of momentum of the liquid coincides with the torsional rigidity of the prism, and the velocity of the liquid at any point represents the shear at that point of the prism. In the first form of the analogy the shear is represented by the velocity of the liquid relative to the vessel.

A result of considerable importance[1] can be at once deduced from the hydrodynamical analogy. Suppose a shaft transmitting a couple to contain a cylindrical flaw whose axis is parallel to that of the shaft. If the diameter of the cavity be small compared with that of the shaft, and the cavity be at a distance from the surface great compared with its diameter, the problem is very nearly the same as that of liquid streaming past a cylinder. Now we know that the velocity of liquid streaming past a cylinder has a maximum value equal to twice the velocity of the stream, and we may therefore conclude that in the case of the shaft the shear near the cavity is twice as great as that at a distance. The importance of this result in connexion with the strength of materials has been previously pointed out (art. 58).

96. Strength of a beam under torsion.

With our notation the six strains of the material are

$$e=f=g=c=0, \quad a=\frac{\partial\phi}{\partial y}+\tau x, \quad b=\frac{\partial\phi}{\partial x}-\tau y,$$

and, by art. 10 (iv), these are equivalent to a shear $\sqrt{(a^2+b^2)}$. This, as stated in the last article, is the velocity of the fluid in the corresponding circulating motion. Whatever theory of strength be adopted it is clearly necessary that this shear must not surpass a certain limit, and it is therefore of great importance to know where the maximum of this shear is to be found. In a number of particular cases it can be shewn that this maximum is on the contour at the point (or points) nearest the axis. To investigate the question generally, it would be necessary to know the solution of the torsion-problem, or of the corresponding hydrodynamical

[1] Larmor, *Phil. Mag.* Jan. 1892.

problem. We can however give reasons for thinking that the points of maximum shear in general lie on the contour and not far from the points nearest the axis. Take the hydrodynamical problem of the liquid circulating within the boundary, and suppose the stream-lines drawn. Then at any point of a stream-line the velocity is inversely proportional to the distance between consecutive stream-lines. As the stream-lines do not cut themselves or each other, and as near the axis they approximate to circles round the axis, and near the boundary they approximate to the boundary, it is clear that in general they will be closest together in the neighbourhood of a line drawn from the axis to the nearest point of the boundary, and farthest apart in the neighbourhood of a line drawn from the axis to the farthest point of the boundary. Again, since the line-integral of the velocity round any closed curve within the boundary is equal to twice the surface-integral of vortex-strength over the area enclosed by the curve, it is clear that the average velocity along any stream-line increases as we pass from the axis to the boundary. This of course does not amount to a proof that the points of maximum shear in the torsion-problem lie on the boundary, but it goes some way towards making this probable.

According to what has just been said the result for a cylindrical cavity requires a more exact statement. Practically it comes to this. Suppose the cavity meets a section in the point P, the shear in the neighbourhood of P is nearly doubled, provided the distance of P from the surface is at least three or four times the diameter of the cavity. If P be near the axis this is of no importance as a greater shear exists at the surface, but if the distance of P from the axis be at least three or four times its distance from the surface the result becomes important as the maximum shear is considerably increased. The case of a semi-circular groove in the surface can be included by observing that the motion of the fluid in its neighbourhood would be the same as that of fluid streaming past a semi-circular ridge on an infinite plane. The maximum shear may in this case be doubled.

If the boundary have anywhere a sharp corner projecting outwards, the velocity of the fluid at the corner vanishes, and therefore the shear in the torsion-problem is zero at such a corner. If the boundary have a sharp corner projecting inwards, the

velocity is theoretically infinite, and the torsion of a prism with such a section will be accompanied by set in the neighbourhood of the corner.

The resistance of the beam to torsion is in general not increased by the existence of a sharp corner projecting outwards, and by paring down the section in the neighbourhood of the corner a form could be arrived at with a smaller area and the same torsional rigidity. The fibres that might be pared away are called by Saint-Venant "useless fibres" (*fibres inutiles*).

97. The circular cylinder.

The boundary condition becomes

$$\psi = \text{const.},$$

so that ψ is constant at all points. Hence also ϕ is constant, and as ϕ vanishes with x and y the value of ϕ is everywhere zero.

In this case the twisting couple is $\mu\tau I$ and there is no distortion of the cross-sections[1].

98. The elliptic cylinder.

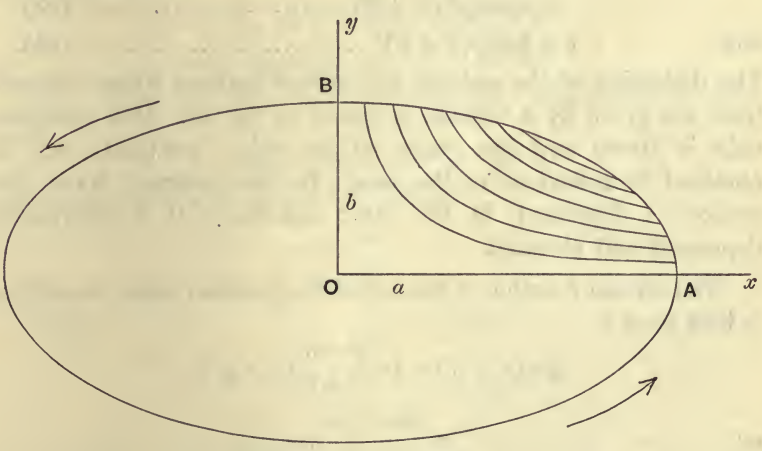

Fig. 12.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49).$$

[1] When the two principal rigidities are unequal there is distortion. Its investigation is left to the reader.

The differential equation is satisfied by

$$\psi = A(x^2 - y^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50),$$

where A is any constant. This will also satisfy the boundary-condition if

$$A(x^2 - y^2) - \tfrac{1}{2}\tau(x^2 + y^2)$$

can be constant when $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,$

which requires that

$$(A - \tfrac{1}{2}\tau)a^2 + (A + \tfrac{1}{2}\tau)b^2 = 0.$$

Hence $$\psi = \tfrac{1}{2}\tau \frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2) \ldots\ldots\ldots\ldots\ldots (51),$$

and $$\phi = -\tau \frac{a^2 - b^2}{a^2 + b^2} xy \ldots\ldots\ldots\ldots\ldots\ldots\ldots (52).$$

In this case the twisting couple $q\mu\tau I$ is

$$\mu\tau I + \mu\tau \iint \frac{a^2 - b^2}{a^2 + b^2}(y^2 - x^2)\, dx\, dy$$

$$= \tfrac{1}{4}\pi ab\mu\tau \left[a^2 + b^2 - \frac{(a^2 - b^2)^2}{a^2 + b^2} \right]$$

$$= \pi\mu\tau a^3 b^3/(a^2 + b^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (53),$$

and $$q = 4a^2 b^2/(a^2 + b^2)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (54).$$

The distortion of the sections into curved surfaces whose contour-lines are given by $\phi = \text{const.}$ is shewn in fig. 12. One quadrant only is drawn and the curves in the other quadrants can be obtained by reflexions in the axes. In the quadrant drawn the section is depressed, in the other quadrants it is alternately depressed and elevated.

The stream-function of the circulating motion when the ellipse is held fixed is

$$\tfrac{1}{2}\omega(x^2 + y^2) - \tfrac{1}{2}\omega \frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2),$$

or $$\omega \frac{x^2 b^2 + y^2 a^2}{a^2 + b^2},$$

the stream-lines are therefore similar ellipses, and the distance between consecutives is proportional to the perpendicular from the centre on the tangent. The maximum velocity along a given stream-line has place therefore on the minor axis. The velocity at any point on the minor axis is proportional to the distance from the centre, and therefore, in the torsion-problem, the shear is

greatest at the extremities of the minor axis of the ellipse, and is there equal to $2\tau a^2 b/(a^2+b^2)$.

99. The equilateral triangular prism.

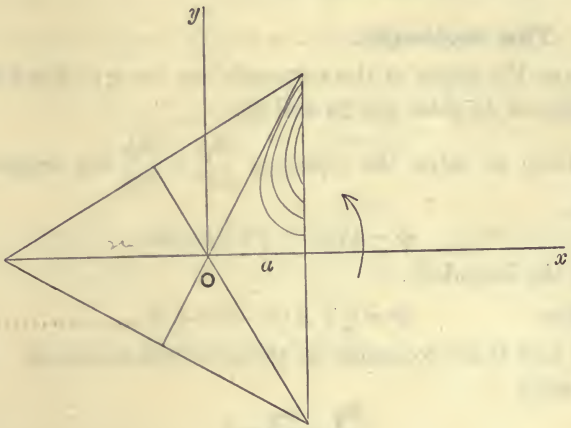

Fig. 13.

Let $3a$ be the height of the triangle, then the equation of the boundary is

$$(x-a)(x-y\sqrt{3}+2a)(x+y\sqrt{3}+2a)=0,$$

or
$$x^3-3xy^2+3a(x^2+y^2)-4a^3=0.$$

The function $A(x^3-3xy^2)$ satisfies the differential equation, and

$$A(x^3-3xy^2)-\tfrac{1}{2}\tau(x^2+y^2)$$

will be constant all over the boundary if

$$A=-\frac{\tau}{6a}.$$

Thus
$$\psi=-\frac{\tau}{6a}(x^3-3xy^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(55),$$

and
$$\phi=-\frac{\tau}{6a}(y^3-3x^2y) \quad \ldots\ldots\ldots\ldots\ldots\ldots(56).$$

The curves of equal distortion are shown in fig. 13. The triangle is divided into 6 exactly similar and equal parts by the lines joining the corners to the middle points of the sides, and, in these parts, the section is alternately depressed and elevated.

In this, as in any case where there is a sharp corner which is not re-entrant, the velocity of the fluid circulating in the prism is zero at the corner, and it is obvious from symmetry that its

maximum in the case of the triangle has place at the middle points of the sides. It may be verified by direct calculation that the maximum shear in the twisted triangular prism has place at the middle points of the sides.

100. The rectangle.

Suppose the edges of the rectangle are $x = \pm a$, $y = \pm b$, so that the lengths of its sides are $2a$ and $2b$.

We have to solve the equation $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$, subject to the condition

$$\psi - \tfrac{1}{2}\tau(x^2 + y^2) = \text{const.}$$

all along the boundary.

Suppose $$\psi = \chi + A(x^2 - y^2) + B \quad \ldots\ldots\ldots\ldots\ldots\ldots (57),$$

where A and B are constants at present undetermined. Then we have to solve

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (58),$$

subject to the condition

$$\chi + A(x^2 - y^2) - \tfrac{1}{2}\tau(x^2 + y^2) + B = 0 \quad \ldots\ldots\ldots\ldots (59)$$

along the edges. If we suppose $a > b$, and take $A = \frac{1}{2}\tau$, $B = b^2\tau$, the conditions for χ become

$$\chi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (60),$$

when $y = \pm b$ and $a > x > -a$, and

$$\chi = \tau(y^2 - b^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (61),$$

when $x = \pm a$ and $b > y > -b$.

The most general form of solution can be expanded in series of the form

$$\sum_m [Ae^{m(x+iy)} + Be^{m(x-iy)}],$$

where the A and B are complex constants. But we observe that χ is an even function of y at the boundaries, and vanishes when $y = \pm b$. Thus we must expand χ in cosines of multiples of y, and make every cosine vanish for $y = b$. Again χ is an even function of x, and thus the x coefficients must all be hyperbolic cosines.

Thus χ is of the form

$$\chi = \sum_0^\infty \left[A_n \frac{\cosh\frac{(2n+1)\pi x}{2b}}{\cosh\frac{(2n+1)\pi a}{2b}} \cos\frac{(2n+1)\pi y}{2b} \right] \quad \ldots\ldots (62),$$

provided

$$\tau(y^2-b^2)=\sum_0^\infty A_n\cos\frac{(2n+1)\pi y}{2b} \quad \text{.....(63),}$$

when $b>y>-b.$

Now between these limits we find, by expanding y^2-b^2 in Fourier's series,

$$A_n=-4\tau b^2\left(\frac{2}{\pi}\right)^3\frac{(-1)^n}{(2n+1)^3} \quad \text{.....(64).}$$

Thus

$$\chi=-4\tau b^2\left(\frac{2}{\pi}\right)^3\sum_0^\infty\left[\frac{(-1)^n}{(2n+1)^3}\frac{\cosh\frac{(2n+1)\pi x}{2b}}{\cosh\frac{(2n+1)\pi a}{2b}}\cos\frac{(2n+1)\pi y}{2b}\right]\text{...(65).}$$

Also $\psi=b^2+\frac{1}{2}\tau(x^2-y^2)+\chi,$

so that

$$\phi=-\tau xy$$
$$+4\tau b^2\left(\frac{2}{\pi}\right)^3\sum_0^\infty\left[\frac{(-1)^n}{(2n+1)^3}\frac{\sinh\frac{(2n+1)\pi x}{2b}}{\cosh\frac{(2n+1)\pi a}{2b}}\sin\frac{(2n+1)\pi y}{2b}\right]\text{...(66).}$$

The twisting couple is, by (39) of art. 93,

$$4\mu\tau ab\frac{a^2+b^2}{3}-\mu\tau\iint(x^2-y^2)\,dx\,dy$$
$$+4\mu\tau b^2\left(\frac{2}{\pi}\right)^3\iint\left(x\frac{\partial\Phi}{\partial y}-y\frac{\partial\Phi}{\partial x}\right)dx\,dy,$$

which is equal to

$$\tfrac{8}{3}\mu\tau ab^3+4\mu\tau b^2\left(\frac{2}{\pi}\right)^3\iint\left(x\frac{\partial\Phi}{\partial y}-y\frac{\partial\Phi}{\partial x}\right)dx\,dy\text{.....(67),}$$

where

$$\Phi=\sum_0^\infty\frac{(-1)^n}{(2n+1)^3}\frac{\sinh\frac{(2n+1)\pi x}{2b}\sin\frac{(2n+1)\pi y}{2b}}{\cosh\frac{(2n+1)\pi a}{2b}}\text{...(68).}$$

A term of the double integral is

$$\operatorname{sech}\frac{(2n+1)\pi a}{2b}\frac{(-1)^n}{(2n+1)^2}\frac{\pi}{2b}\iint\left[x\sinh\frac{(2n+1)\pi x}{2b}\cos\frac{(2n+1)\pi y}{2b}\right.$$
$$\left.-y\cosh\frac{(2n+1)\pi x}{2b}\sin\frac{(2n+1)\pi y}{2b}\right]dx\,dy.$$

Now

$$\int_{-a}^{a} x \sinh \frac{(2n+1)\pi x}{2b}\, dx$$

$$= \frac{2b}{(2n+1)\pi}\left[2a \cosh \frac{(2n+1)\pi a}{2b} - \frac{2b}{(2n+1)\pi} 2 \sinh \frac{(2n+1)\pi a}{2b}\right],$$

and

$$\int_{-b}^{b} \cos \frac{(2n+1)\pi y}{2b}\, dy = \frac{2b}{(2n+1)\pi} 2(-1)^n,$$

so

$$\int_{-b}^{b} y \sin \frac{(2n+1)\pi y}{2b}\, dy = \frac{8b^2(-1)^n}{(2n+1)^2\pi^2}.$$

Also

$$\int_{-a}^{a} \cosh \frac{(2n+1)\pi x}{2b}\, dx = \frac{2b}{(2n+1)\pi} 2 \sinh \frac{(2n+1)\pi a}{2b}.$$

Hence the twisting couple is

$$\tfrac{8}{3}\mu\tau ab^3 + \left(\frac{4}{\pi}\right)^4 \mu\tau ab^3 \sum_0^\infty \frac{1}{(2n+1)^4}$$

$$- 4\mu\tau b^2 \left(\frac{2}{\pi}\right)^3 \sum_0^\infty \left(\frac{2b}{\pi}\right)^2 \frac{8 \tanh \dfrac{(2n+1)\pi a}{2b}}{(2n+1)^5},$$

which is equal to

$$\tfrac{16}{3}\mu\tau ab^3 - \mu\tau b^4 \left(\frac{4}{\pi}\right)^5 \sum_0^\infty \frac{\tanh \dfrac{(2n+1)\pi a}{2b}}{(2n+1)^5} \quad \ldots\ldots\ldots (69),$$

since

$$\sum_0^\infty \frac{1}{(2n+1)^4} = \frac{\pi^4}{96},$$

and it is to be noticed that a term $\frac{8}{3}\mu\tau ab^3$ is contributed by the transcendental part of ϕ.

The expression for ϕ must really be unaltered when x is changed into y, and a into b, and the expression for the twisting couple must be symmetrical in a and b. For an account of the identities thus obtained the reader is referred to a paper by Mr F. Purser in the *Messenger of Mathematics*, XI. 1882.

Saint-Venant has investigated[1] the forms of the curves of equal distortion given by $\phi = \text{const.}$ If we begin with the case of a square prism ϕ vanishes along the diagonals, and along the middle lines of the square parallel to its sides. If we take one side great compared with the adjacent side, then ϕ vanishes only on the middle lines and not on the diagonals. The limiting form between

[1] See the great memoir on 'Torsion' of 1855.

rectangles which divide into 8 parts, in which ϕ is alternately positive and negative, and those which divide into 4 such parts, is given by making the ratio of the sides equal to 1·4513.

101. Other sections.

When the section is not of one of the forms just considered the problem can frequently be solved by means of conjugate functions. Whenever we know a transformation by means of conjugate functions, say

$$\alpha + \iota\beta = f(x + \iota y),$$

such that the boundary consists partly of lines $\alpha = \text{const.}$ and partly of lines $\beta = \text{const.}$, the differential equation for ψ can be expressed in the form

$$\frac{\partial^2\psi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\beta^2} = 0,$$

and ψ will be a given function of α along some part of the boundary, and of β along the remainder. The simplest case is that of a curvilinear rectangle bounded by two arcs of concentric circles and two radii, including the case of a sector[1] of a circle of any angle. The work in these cases may be left to the reader, we give the results.

1°. For a sector of angle 2β in terms of polar coordinates r, θ, so that the boundaries are

$$r = 0, \quad r = a, \quad \theta = \pm\beta,$$

$$\psi = \tfrac{1}{2}\tau r^2 \frac{\cos 2\theta}{\cos 2\beta} + \tau a^2 \sum_0^\infty \left[A_{2n+1} \left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\beta}} \cos(2n+1)\frac{\pi\theta}{2\beta} \right] \quad \ldots(70),$$

where

$$A_{2n+1} = (-)^{n+1} \left[\frac{1}{(2n+1)\pi - 4\beta} - \frac{2}{(2n+1)\pi} + \frac{1}{(2n+1)\pi + 4\beta} \right] \quad \ldots\ldots\ldots(71).$$

If we write $re^{\iota\theta} = ax$, then

$$\psi - \iota\phi = \tfrac{1}{2}\tau a^2 \frac{x^2}{\cos 2\beta}$$

$$- \frac{\tau a^2}{2\beta} \left\{ x^2 \int_0^x \frac{x^{\frac{\pi}{2\beta}-3}}{1 + x^{\frac{\pi}{\beta}}}\, dx - \frac{4\beta}{\pi} \tan^{-1}\left(x^{\frac{\pi}{2\beta}}\right) + \frac{1}{x^2} \int_0^x \frac{x^{\frac{\pi}{2\beta}+1}}{1 + x^{\frac{\pi}{\beta}}}\, dx \right\} \quad \ldots(72),$$

where the modulus of x is $\not> 1$, and $\tan^{-1}\left(x^{\frac{\pi}{2\beta}}\right)$ is that branch of the function which vanishes with x.

[1] Greenhill, *Messenger of Mathematics*, VIII. 1879, and X. 1881.

2°. In case $\frac{\pi}{2\beta}$ is an integer greater than 2 the integrations can be performed, but when $\frac{\pi}{2\beta}=2$ the first two terms become infinite, and their difference has a finite limit, and we find for a quadrantal cylinder

$$\psi-\iota\phi=\frac{2\tau a^2}{\pi}\left[-x^2\log x+\tan^{-1}x^2+\tfrac{1}{4}\left(x^2-\frac{1}{x^2}\right)\log(1+x^4)\right] \quad \text{.....(73).}$$

3°. For a semi-circular cylinder

$$\psi-\iota\phi=\frac{\tau a^2}{\pi}\left[\tfrac{1}{2}\pi x^2-i\left(x+\frac{1}{x}\right)+\tfrac{1}{2}\iota\left(x^2+\frac{1}{x^2}-2\right)\log\frac{1+x}{1-x}\right] \quad \text{...(74).}$$

4°. For a curvilinear rectangle bounded by two arcs and two radii, taking conjugate functions

$$x+\iota y=ce^{\alpha+\iota\beta} \quad \text{.....(75),}$$

and supposing the outer radius is $a=ce^{\alpha_0}$, and the inner $b=ce^{-\alpha_0}$, (so that c is the geometrical mean of the radii,) and taking for the bounding radii $\beta=\pm\beta_0$, we find

$$\phi=-\tfrac{1}{2}\tau abe^{2\alpha}\frac{\sin 2\beta}{\cos 2\beta_0}+2^5\tau ab\beta_0{}^2\sum_0^\infty A_n\Phi_n,$$

where

$$\Phi_n=\left\{\cosh 2\alpha_0\frac{\sinh\frac{(2n+1)\pi\alpha}{2\beta_0}}{\cosh\frac{(2n+1)\pi\alpha_0}{2\beta_0}}+\sinh 2\alpha_0\frac{\cosh\frac{(2n+1)\pi\alpha}{2\beta_0}}{\sinh\frac{(2n+1)\pi\alpha_0}{2\beta_0}}\right\} \quad \text{...(76),}$$

and

$$A_n=\frac{(-)^n\sin\frac{(2n+1)\pi\beta}{2\beta_0}}{\{(2n+1)\pi-4\beta_0\}(2n+1)\pi\{(2n+1)\pi+4\beta_0\}}.$$

Another method is to take any function ψ satisfying

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0,$$

and make $$\psi-\tfrac{1}{2}\tau(x^2+y^2)=\text{const.},$$

the equation of a bounding curve. Then this gives a boundary for which the problem of torsion is solved.

102. Approximate Formula.

A large number of such cases have been solved, and the results obtained lead to a very remarkable approximate formula for the

torsion of a prism of any section which is not very elongated in any direction, and does not present any re-entrant angles.

This formula is a generalisation of the formula for the ellipse. The latter gives for the twisting couple G the form

$$G = \frac{1}{4\pi^2} \frac{\omega^4}{I} \mu\tau,$$

where ω is the area and I the moment of inertia about the axis.

Now Saint-Venant[1] found that, in the case of all sections such as those just described that had been worked out,

$$G = \kappa \frac{\omega^4}{I} \mu\tau,$$

where the value of κ varies only from ·0228 to ·026 while its mean value is about ·025, or $\frac{1}{40}$, and its value for an ellipse is ·02533..., and thus we may take with remarkable accuracy for most forms of section likely to occur in practice

$$G = \tfrac{1}{40} \frac{\omega^4}{I} \mu\tau \quad \text{.....(77),}$$

and the twisting couple for different prisms of the same material is directly as the fourth power of the area of the cross-section, and inversely as the moment of inertia of this section about the axis. The theory of Coulomb and Navier made the couple directly proportional to the moment of inertia I.

103. Torsion of Æolotropic Rectangular Beam.

On account of its importance we shall give the solution of the problem of the torsion of the rectangular prism in the case where the two principal rigidities of the material for shear of planes through the axis are not the same.

We have to find a function ϕ to satisfy the differential equation

$$M \frac{\partial^2 \phi}{\partial x^2} + L \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{.....(78),}$$

and the conditions

$$\frac{\partial \phi}{\partial x} = \tau y, \text{ when } x = \pm a, \text{ and } b > y > -b,$$

and

$$\frac{\partial \phi}{\partial y} = -\tau x, \text{ when } y = \pm b \text{ and } a > x > -a.$$

[1] *Comptes Rendus*, LXXXVIII. pp. 142—147, 1879.

Let
$$\left.\begin{aligned} x &= x'\sqrt{M}/\sqrt{(L+M)} \\ y &= y'\sqrt{L}/\sqrt{(L+M)} \end{aligned}\right\} \quad \text{.....(79),}$$

where the denominator is introduced for the sake of homogeneity.

The differential equation (78) becomes

$$\frac{\partial^2\phi}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} = 0 \quad \text{.....(80),}$$

and the boundary-conditions become

$$\frac{\partial\phi}{\partial x'} = \frac{\sqrt{(LM)}}{L+M}\tau y' \quad \text{.....(81),}$$

when
$$x' = \pm\sqrt{\left(\frac{L+M}{M}\right)}a$$

and
$$\sqrt{\left(\frac{L+M}{L}\right)}b > y' > -\sqrt{\left(\frac{L+M}{L}\right)}b,$$

and
$$\frac{\partial\phi}{\partial y'} = -\frac{\sqrt{(LM)}}{L+M}\tau x' \quad \text{.....(82),}$$

when
$$y' = \pm\sqrt{\left(\frac{L+M}{L}\right)}b$$

and
$$\sqrt{\left(\frac{L+M}{M}\right)}a > x' > -\sqrt{\left(\frac{L+M}{M}\right)}a.$$

Introduce ψ the conjugate function of ϕ with respect to x' and y', supposing $\phi + \iota\psi$ to be a function of $x' + \iota y'$; then, writing

$$\left.\begin{aligned} &\frac{\sqrt{(LM)}}{L+M}\tau = \tau', \qquad \sqrt{\left(\frac{L+M}{M}\right)}a = a', \\ &\sqrt{\left(\frac{L+M}{L}\right)}b = b' \end{aligned}\right\} \quad \text{.....(83),}$$

ψ has to satisfy the conditions

$$\frac{\partial\psi}{\partial y'} = \tau' y' \quad \text{.....(84),}$$

when
$$x' = \pm a' \text{ and } b' > y' > -b',$$

and
$$\frac{\partial\psi}{\partial x'} = \tau' x' \quad \text{.....(85),}$$

when
$$y' = \pm b' \text{ and } a' > x' > -a'.$$

Thus
$$\psi = \tfrac{1}{2}\tau'(x'^2 + y'^2) + \text{const.} \quad \text{.....(86)}$$

is the condition that obtains at the boundary.

The problem is now precisely the same as in the case of elastic symmetry except that

$$x',\ y',\ a',\ b',\ \tau'$$

take the places of

$$x,\ y,\ a,\ b,\ \tau.$$

We have therefore

$$\phi = -\tau' x' y' + \tau' \frac{2^5 b'^2}{\pi^3} \sum_0^\infty \left[\frac{(-)^n}{(2n+1)^3} \frac{\sinh \frac{(2n+1)\pi x'}{2b'}}{\cosh \frac{(2n+1)\pi a'}{2b'}} \sin \frac{(2n+1)\pi y'}{2b'} \right],$$

or

$$\phi = -\tau x y + \tau \sqrt{\left(\frac{M}{L}\right)} \frac{2^5 b^2}{\pi^3} \left[\sum_0^\infty \frac{(-)^n}{(2n+1)^3} \frac{\sinh \left\{ \frac{(2n+1)\pi x}{2b} \sqrt{\frac{L}{M}} \right\}}{\cosh \left\{ \frac{(2n+1)\pi a}{2b} \sqrt{\frac{L}{M}} \right\}} \sin \frac{(2n+1)\pi y}{2} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (87).$$

And the twisting couple is, by (38),

$$\tau (LI_1 + MI_2) + \iint \left\{ Lx \frac{\partial \phi}{\partial y} - My \frac{\partial \phi}{\partial x} \right\} dx dy,$$

which can be shewn to be equal to

$$\frac{16}{3} M\tau a b^3 - M \sqrt{\left(\frac{M}{L}\right)} \tau b^4 \left(\frac{4}{\pi}\right)^5 \sum_0^\infty \left[\frac{\tanh \left\{ \frac{(2n+1)\pi a}{2b} \sqrt{\left(\frac{L}{M}\right)} \right\}}{(2n+1)^5} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (88).$$

104. Approximate Formula.

We may express this result in the following form.

Let l be the length of the beam, t its thickness (or smaller cross-dimension), ω the area of its cross-section, r the ratio $(a\sqrt{L})/(b\sqrt{M})$, f the function

$$\left(\frac{4}{\pi}\right)^5 \sum_0^\infty \frac{\tanh \frac{(2n+1)\pi r}{2}}{(2n+1)^5} \quad \ldots\ldots\ldots\ldots\ldots\ldots (89).$$

Then the angle turned through by the end at which the couple G is applied is

$$\frac{3Gl}{M\omega t^2 \left(1 - \frac{3}{16} f/r\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (90).$$

Now when r is not < 3, f is remarkably nearly constant and equal

to 3·361...; and thus, if the thickness of the beam be considerably less than the breadth, we have an approximate formula

$$\text{angle turned through} = \frac{3Gl}{M\omega t^2\left(1 - \frac{\cdot 63}{r}\right)}.$$

In this formula M is the rigidity for the directions of the axis of the beam and the breadth of the cross-section, supposed perpendicular to planes of symmetry of the material.

It is clear that by twisting thin rectangular bars of a crystal, cut parallel to axes of symmetry, and having their smaller sectional dimensions parallel to axes of symmetry, we can obtain sufficient data to determine the three principal rigidities of the crystal. Prof. Voigt of Göttingen has used this method to determine the rigidities of certain crystals. He has also shewn that a formula similar to (90) holds in the case of other crystal forms than those having three planes of symmetry, only M is not the rigidity for the two directions, and f is a function which satisfies equations that have not been solved. At present we refer the reader to Prof. Voigt's papers[1] for an account of the torsion of prisms of matter which has not three planes of symmetry of contexture, and we hope to return to the subject in connexion with Kirchhoff's theory of thin rods.

105. Non-uniform Flexure.

Returning now to the general solutions of art. 88, we wish to shew how they may be applied to the case of flexure by transverse force applied at one end of the beam. For this purpose we must consider the terms of the solution which involve the constants β_1 and β_2. We have seen already in art. 89 that the solution involving β_2 is distinct from that involving β_1 and that one of these can be obtained from the other, and it will therefore be sufficient if we consider the terms in β_1 only.

Suppose then that all the constants in equations (23) except β_1 vanish, and let us seek to determine the end-traction that gives rise to the terms in β_1.

The displacements are

$$\left.\begin{aligned} u &= -\tfrac{1}{2}\beta_1 z\,(\sigma_1 x^2 - \sigma_2 y^2) - \tfrac{1}{6}\beta_1 z^3 \\ v &= -\sigma_2 \beta_1 xyz \\ w &= \phi + \tfrac{1}{2}z^2\beta_1 x - \tfrac{1}{2}\,(E - M\sigma_1 - L\sigma_2)\,\beta_1 xy^2/L \end{aligned}\right\} \quad \ldots\ldots(91),$$

[1] For references see p. 146 *supra*.

where ϕ satisfies the differential equation $M\frac{\partial^2\phi}{\partial x^2}+L\frac{\partial^2\phi}{\partial y^2}=0$ ...(92), at all points of a normal section, and the condition

$$Ml\frac{\partial\phi}{\partial x}+Lm\frac{\partial\phi}{\partial y}$$
$$=\beta_1\left[\tfrac{1}{2}Ml\sigma_1x^2+\frac{E-M\sigma_1}{L}(Lmxy+\tfrac{1}{2}Mly^2)-Ml\sigma_2y^2\right] \quad \text{.....(93)}$$

at all points of the boundary.

The stresses at any point of a section, $z=$ const., are

$$\left.\begin{aligned} S&=L\left[\frac{\partial\phi}{\partial y}-\frac{E-M\sigma_1}{L}\beta_1xy\right] \text{ parallel to the axis } y,\\ T&=M\left[\frac{\partial\phi}{\partial x}-\tfrac{1}{2}\beta_1\left(\sigma_1x^2+\frac{E-M\sigma_1-2L\sigma_2}{L}y^2\right)\right]\\ &\qquad\qquad \text{parallel to the axis } x,\\ R&=E\beta_1zx \text{ parallel to the axis } z \end{aligned}\right\} \text{...(94).}$$

The resultant stress parallel to x is

$$M\iint\left[\frac{\partial\phi}{\partial x}-\tfrac{1}{2}\beta_1\left(\sigma_1x^2+\frac{E-M\sigma_1-2L\sigma_2}{L}y^2\right)\right]dxdy$$
$$=\iint\left[M\frac{\partial}{\partial x}\left(x\frac{\partial\phi}{\partial x}\right)+L\frac{\partial}{\partial y}\left(x\frac{\partial\phi}{\partial y}\right)\right]dxdy$$
$$-\tfrac{1}{2}\beta_1M\iint\left(\sigma_1x^2+\frac{E-M\sigma_1-2L\sigma_2}{L}y^2\right)dxdy.$$

The first surface-integral on the right can be transformed into a line-integral round the boundary, and, if ds be an element of arc of the boundary, and l, m the direction-cosines of the normal to it drawn outwards, the stress parallel to x becomes, by (93),

$$\beta_1\int\left[\tfrac{1}{2}Ml\sigma_1x^3+\frac{E-M\sigma_1}{L}(Lmx^2y+\tfrac{1}{2}Mly^2x)-Ml\sigma_2y^2x\right]ds$$
$$-\tfrac{1}{2}\beta_1M\iint\left(\sigma_1x^2+\frac{E-M\sigma_1-2L\sigma_2}{L}y^2\right)dxdy.$$

Transforming the line-integral into a surface-integral, we find stress parallel to $x=EI_1\beta_1$.....(95), with our previous notation for the moments of inertia.

The resultant stress parallel to y is

$$L\iint\left[\frac{\partial\phi}{\partial y}-\frac{E-M\sigma_1}{L}\beta_1xy\right]dxdy$$
$$=\iint\left[L\frac{\partial}{\partial y}\left(y\frac{\partial\phi}{\partial y}\right)+M\frac{\partial}{\partial x}\left(y\frac{\partial\phi}{\partial x}\right)\right]dxdy-\beta_1\iint(E-M\sigma_1)xydxdy.$$

Transforming as before we find that this is

$$\beta_1 \int \left\{ \tfrac{1}{2} M l \sigma_1 x^2 y + \frac{E - M\sigma_1}{L} [Lmxy^2 + \tfrac{1}{2} M l y^3] \right\} ds$$

$$- \beta_1 \iint (E - M\sigma_1)\, xy dx dy$$

$$= \beta_1 \iint E xy dx dy,$$

which vanishes provided the principal axes of inertia of the section of the beam coincide with its axes of symmetry of elastic structure.

The resultant stress parallel to z vanishes.

The couple about an axis in the section parallel to the axis x is

$$z\beta_1 E \iint xy dx dy,$$

which vanishes identically when the stress parallel to y vanishes.

The couple about an axis in the section parallel to the axis y is

$$- z\beta_1 E \iint x^2 dx dy = - z\beta_1 E . I_1 \quad \ldots\ldots\ldots\ldots (96).$$

The couple about the axis z at any section is

$$\iint \left(Lx \frac{\partial \phi}{\partial y} - My \frac{\partial \phi}{\partial x} \right) dx dy$$

$$- \beta_1 M \iint [(E - M\sigma_1)(x^2 y / M - \tfrac{1}{2} y^3 / L) - \tfrac{1}{2} \sigma_1 x^2 y + \sigma_2 y^3]\, dx dy \ldots (97),$$

and, as in the case of torsion, the first term of this can only be calculated when ϕ is known.

We shall suppose that the principal axes of inertia of the section of the beam coincide with its axes of symmetry of elastic structure, and then we see that the resultant stress at any section reduces to a transverse force parallel to the axis x, a bending couple about the axis y, and a torsional couple about the axis z. By a combination of this solution with the previous solutions, in which there was either simply a bending couple about the axis of y, or simply a torsional couple about the axis of z, we may obtain the solution for any resultant stress-system.

106. Practical Utility of the Solution.

Now although the resultant stress-system may consist of any extending force, any transverse force, any twisting couple, and any bending couple whatever, yet the mode found for the application of these forces is unique and determinate. It is very unlikely

that in the pieces used in architecture or engineering the stresses applied at an end of a bar are distributed over the end in the manner here considered, and the application of the theory to structures depends on the validity of a principle introduced by Saint-Venant[1] which may be called the principle of the "elastic equivalence of statically equipollent loads". He stated that *any system of surface-tractions applied to a small part of a surface of a solid will produce at any point not very near to that surface the same strain as any other system of forces having the same resultant would produce.* When the part of the surface to which the forces are applied is vanishingly small the principle is obviously true, but it becomes less and less exact as the surface subjected to traction becomes larger. The principle is in fact equivalent to this—the application of an equilibrating system of forces to a part of the surface of a solid produces no strain in the interior—a theorem which is in general obviously untrue, but which becomes more and more exact as the surface subjected to traction is diminished. It is on account of the assumed smallness of the linear dimensions of the cross-section of the beam compared with its length that the principle may be applied to obtain a theory of the equilibrium of beams.

107. Bending by transverse force.

We wish to obtain in the general case the solution for applied transverse force X parallel to the axis x.

Consider first the terms in β_1, these give rise to (1) a stress $E\beta_1 I_1$ parallel to the axis x, (2) a couple $-E\beta_1 I_1 z$ about the axis y, and (3) a couple which we may call T about the axis z. (See art. 97.) T is constant along the beam.

The terms in α_1 give rise to a couple $-E\alpha_1 I_1$ about the axis y. (See art. 89.)

The terms in β_0 or τ give rise to a couple about the axis z (see art. 92), and since τ is at our disposal we may make this couple equal to $-T$. This couple is constant all along the beam.

If then we take $\alpha_1 + \beta_1 l = 0$, the stress-system at $z = l$ will reduce to a force X where

$$X = E\beta_1 I_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (98).$$

[1] See the memoir on 'Torsion' of 1855.

The stress at any other section will reduce to a force X and a couple

$$E\beta_1 I_1 (l-z) \quad \text{.....(99).}$$

Since

$$\alpha_1 = -\beta_1 l = -Xl/EI_1,$$

the equation of the central-line is by (23)

$$x = \frac{X}{EI_1}\left(\tfrac{1}{2}z^2 l - \tfrac{1}{6}z^3\right),$$

and the stress-system consists of a force parallel to the axis x

$$-EI_1 \frac{d^3x}{dz^3} \quad \text{.....(100),}$$

and a couple about the axis y

$$EI_1 \frac{d^2x}{dz^2} \quad \text{.....(101).}$$

The quantity EI_1 is called the *flexural rigidity* of the beam, and the above analysis verifies the ordinary theory of the "bending moment", viz.: that the flexural couple at any point of a slightly bent beam is the product of the curvature and the flexural rigidity.

Let AB be the beam, P any point on its central-line, then the system of forces just considered is the resultant of the action of

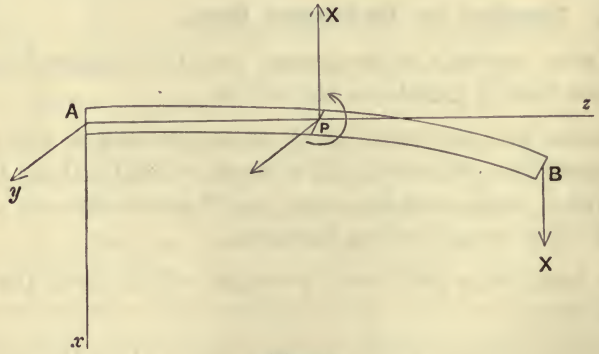

Fig. 14.

the part BP on the part AP, and the action of AP on BP is equal and opposite to this.

Now it may be observed that the equation

$$EI_1 \frac{d^2x}{dz^2} = X(l-z) \quad \text{.....(102),}$$

which is derived at once from the figure by taking moments for

the equilibrium of BP about an axis through P parallel to the axis y, is sufficient, with the conditions

$$x = 0, \frac{dx}{dz} = 0$$

at $z = 0$, to determine the form of the central-line.

We shall devote a subsequent chapter to the development of this remark.

At present we notice that the deflexion of the axis of the beam is

$$\tfrac{1}{2} \frac{X}{EI_1} (l - \tfrac{1}{3}z) z^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (103),$$

and the maximum deflexion is at the end of the beam, and is equal to

$$\tfrac{1}{3} \frac{Xl^3}{EI_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (104).$$

The displacements are

$$\left.\begin{aligned} u &= \tfrac{1}{2} \frac{X}{EI_1} [(l - z)(\sigma_1 x^2 - \sigma_2 y^2) + (l - \tfrac{1}{3}z) z^2] - \tau yz \\ v &= \frac{X}{EI_1} \sigma_2 (l - z) xy + \tau xz \\ w &= \tfrac{1}{2} \frac{X}{EI_1} \left[(z^2 - 2lz) x - \frac{E - M\sigma_1 - L\sigma_2}{L} xy^2 \right] + \phi \end{aligned}\right\} \ldots\ldots (105),$$

where τ is the twist corresponding to the twisting couple $-T$.

The changes in any cross-section may be analysed into:—

(1) A rigid-body translational displacement parallel to x which brings the centre of inertia to its proper place on the bent axis of the beam.

(2) A rigid-body rotational displacement τz round the axis of the beam[1].

(3) A distortion of the shape of the cross-section similar to that sketched in art. 91, the displacements involved being proportional to the distance from the end at which the force X is applied.

(4) A distortion of the cross-sections into curved surfaces depending on the function ϕ.

Readers of Saint-Venant's Memoir of 1856 will see that his expression for the displacement u contains an additional term

[1] In all the particular cases that we shall treat τ vanishes.

proportional to z, and in fact equivalent to $-z\left(\frac{\partial w}{\partial x}\right)_0$, where the suffix 0 refers to the value at the origin, this cannot occur in our solution because we have imposed the geometrical condition that $\left(\frac{\partial u}{\partial z}\right)_0 = 0$. Saint-Venant consequently obtains an additional term $-l\left(\frac{\partial w}{\partial x}\right)_0$ in the maximum deflexion, and Prof. Pearson[1] appears to think that this constitutes a correction (though not a very important one) to the Bernoulli-Eulerian theory of beams. As a matter of fact it only amounts to superposing on the displacements (105) a certain rigid-body rotation, and need not therefore be considered.

108. Asymmetric Loading.

Suppose the principal axes of inertia of the section of the beam do not lie in and perpendicular to the plane through the axis z and the direction of the force. Then, still supposing the axes of x and y to be these principal axes, the force will have two components X and Y.

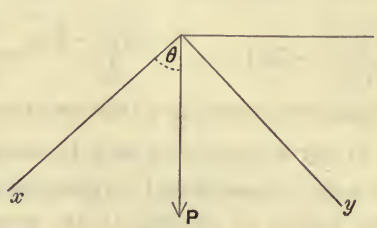

Fig. 15.

Let P be the applied force, θ the angle between the direction of P and the axis of x, then $P\cos\theta = X$, and $P\sin\theta = Y$, and we have to add the solutions for

$$\left.\begin{aligned} \beta_1 &= \frac{P\cos\theta}{EI_1}, \quad \beta_2 = \frac{P\sin\theta}{EI_2} \\ \alpha_1 &= -\frac{Pl\cos\theta}{EI_1}, \quad \alpha_2 = -\frac{Pl\sin\theta}{EI_2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(106).$$

Let ϕ_1 be the value of ϕ corresponding to β_1, and let ϕ_2 be the value of ϕ corresponding to β_2, then the twisting couple T is the sum of the two twisting couples that come from ϕ_1 and ϕ_2.

[1] *Elastical Researches of Barré de Saint-Venant*, art. 96.

The displacements for bending by the force P are, by 105,

$$\left.\begin{aligned} u &= \tfrac{1}{2}\frac{P\cos\theta}{EI_1}[(l-z)(\sigma_1 x^2-\sigma_2 y^2)+(l-\tfrac{1}{3}z)z^2] \\ &\qquad +\frac{P\sin\theta}{EI_2}(l-z)\sigma_1 xy-\tau yz, \\ v &= \tfrac{1}{2}\frac{P\sin\theta}{EI_2}[(l-z)(\sigma_2 y^2-\sigma_1 x^2)+(l-\tfrac{1}{3}z)z^2] \\ &\qquad +\frac{P\cos\theta}{EI_1}(l-z)\sigma_2 xy+\tau xz, \\ w &= \tfrac{1}{2}\frac{P\cos\theta}{EI_1}\left[(z^2-2lz)x-\frac{E-M\sigma_1-L\sigma_2}{L}xy^2\right] \\ &\quad +\tfrac{1}{2}\frac{P\sin\theta}{EI_2}\left[(z^2-2lz)y-\frac{E-M\sigma_1-L\sigma_2}{M}yx^2\right] \\ &\quad +\phi_1+\phi_2 \end{aligned}\right\} \quad \ldots\ldots(107),$$

where τ is the twist corresponding to the twisting couple $-T$.

The equations of the central-line are

$$\left.\begin{aligned} x &= \tfrac{1}{2}\frac{P\cos\theta}{EI_1}(l-\tfrac{1}{3}z)z^2 \\ y &= \tfrac{1}{2}\frac{P\sin\theta}{EI_2}(l-\tfrac{1}{3}z)z^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(108),$$

and the deflexion in the plane through the axis and the direction of the force P is

$$\tfrac{1}{2}\frac{P}{E}\left(\frac{\cos^2\theta}{I_1}+\frac{\sin^2\theta}{I_2}\right)(l-\tfrac{1}{3}z)z^2 \ldots\ldots\ldots\ldots(109).$$

The maximum deflexion is at the end of the beam and is equal to

$$\tfrac{1}{3}\frac{Pl^3}{E}\left(\frac{\cos^2\theta}{I_1}+\frac{\sin^2\theta}{I_2}\right) \ldots\ldots\ldots\ldots(110).$$

Unless $I_1 = I_2$ the axis will not be bent into a plane curve in the plane through P and the old position of the axis, but the plane in which the curve lies is given by the equation

$$\frac{x\sin\theta}{I_2}-\frac{y\cos\theta}{I_1}=0;$$

the sine of the angle between this plane and the plane through P and the axis is

$$\sin\theta\cos\theta\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\left(\frac{\sin^2\theta}{I_2^2}+\frac{\cos^2\theta}{I_1^2}\right)^{-\frac{1}{2}}.$$

The plane through the axis perpendicular to the plane in which bending takes place cuts the cross-section in a line which is conjugate, with respect to the ellipse $x^2/I_1 + y^2/I_2 = 1$, to the intersection of the cross-section and the plane through the axis and the direction of the force P.

This is the theorem of Saint-Venant and Bresse that the neutral line is conjugate with respect to the ellipse of inertia to the trace of the load-plane on the cross-section.

109. Strength of a beam under flexure.

To simplify[1] the consideration of this question we shall suppose that $\sigma_1 = \sigma_2 = \sigma$, and $L = M = \mu$; also we shall suppose the section such that the twisting couple $T = 0$, and thus reduce the displacements to

$$\left.\begin{aligned} u &= \tfrac{1}{2}\beta_1 [\sigma (l-z)(x^2 - y^2) + (l - \tfrac{1}{3}z) z^2] \\ v &= \beta_1 \sigma (l-z) xy \\ w &= \tfrac{1}{2}\beta_1 \left[(z^2 - 2lz) x - \left(\frac{E}{\mu} - 2\sigma \right) xy^2 \right] + \phi \end{aligned} \right\} \quad \ldots\ldots(111),$$

where ϕ is a certain function of (x, y) to be determined by solving the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

subject to a condition given at the boundary.

The six strains are

$$\left.\begin{aligned} e &= \beta_1 \sigma (l-z) x \\ f &= \beta_1 \sigma (l-z) x \\ g &= -\beta_1 (l-z) x \\ a &= \frac{\partial \phi}{\partial y} - \left(\frac{E}{\mu} - \sigma \right) \beta_1 xy \\ b &= \frac{\partial \phi}{\partial x} - \tfrac{1}{2}\beta_1 \left[\left(\frac{E}{\mu} - 3\sigma \right) y^2 + \sigma x^2 \right] \\ c &= 0 \end{aligned} \right\} \quad \ldots\ldots(112)$$

Now in the particular cases that we shall investigate we shall find that ϕ is at least of the order 3 in the linear dimensions of the cross-section, and thus, if x and y be small

[1] The suppositions of this article are not equivalent to that of transverse isotropy. See Note B at the end of the volume.

compared with $(l-z)$, we may for a first approximation neglect a and b in comparison with e, f, g. It will then be seen that the greatest principal extension has place at the highest point on the contour and at the fixed end, x being measured positive downwards. According to the theory of Poncelet and Saint-Venant (art. 57) this extension must not exceed a certain limit. If T_0 be the breaking stress of the material for pull in the direction of the axis z, and t the thickness of the beam the limit of safe loading X is given by the equation

$$\Phi X = 2T_0 I_1/(lt) \quad \ldots\ldots\ldots\ldots\ldots\ldots (113),$$

where Φ is a factor of safety.

In case the beam is not very long[1] in comparison with its breadth and thickness, we cannot neglect the shears a, b, due to flexure, and given in equations (112). The problem then becomes very complicated. We have really to transform the elongation quadric

$$(e, f, g, \tfrac{1}{2}a, \tfrac{1}{2}b, \tfrac{1}{2}c)(x, y, z)^2 = k$$

to its principal axes, so as to obtain the form

$$e_1 x^2 + f_1 y^2 + g_1 z^2 = k,$$

then the greatest positive value of e_1, f_1, or g_1 is the greatest principal extension. The quadric can be found in particular cases only after the function ϕ has been determined.

110. Strength under combined strain.

Suppose the strains due to transverse force $X\,(=\beta_1 EI_1)$ to be as in the last article

$$e = \beta_1\sigma\,(l-z)\,x, \quad f = \beta_1\sigma\,(l-z)\,x, \quad g = -\beta_1\,(l-z)\,x,$$

and the strains due to torsional couple G to be as in art. 96

$$a = \frac{\partial\phi}{\partial y} + \tau x, \quad b = \frac{\partial\phi}{\partial x} - \tau y,$$

and suppose $c = 0$.

The beam will be at once bent to curvature $\beta_1\,(l-z)$ at any point, and twisted to torsion τ.

The principal axes of the strain at any point are a line parallel to the axis z and two rectangular lines in the plane of the section. It is easy to see that the latter make half-right angles with the tangent and normal to the stream-line of the circulating fluid

[1] If this be not the case the forces whose resultant is X must be regarded as distributed over the end in a particular manner. See art. 106.

motion associated with the torsional strain by the hydrodynamical analogy of art. 95. For the strains e and f at any point, being equal, are equivalent to uniform extension $\beta_1\sigma(l-z)x$ in the plane $z=$ const., and the principal axes of the shear compounded of a and b are the lines in question. If s be the amount of the shear $\sqrt{(a^2+b^2)}$, then the principal extensions that arise from the torsional strain are $\frac{1}{2}s$ and $-\frac{1}{2}s$, and by superposing these on the uniform extension we find that the principal extensions for the combined torsional and flexural strain are

$$\beta_1\sigma(l-z)x+\tfrac{1}{2}s, \quad \beta_1\sigma(l-z)x-\tfrac{1}{2}s, \quad -\beta_1(l-z)x \ldots (114).$$

The maximum is either the maximum of $-\beta_1(l-z)x$ when x is negative and has its greatest value, or of $\beta_1\sigma(l-z)x+\frac{1}{2}s$ when x is positive. Let us first suppose s very small or that τ is small in comparison with $\beta_1(l-z)$, then the limit of safe loading is determined by considering the flexure and disregarding the torsion. In the same way if τ be great in comparison with $\beta_1(l-z)$ the limit of safe-loading is determined by considering the torsion and disregarding the flexure.

In general there is an abrupt change from circumstances in which safety is determined purely by considerations of flexure to others in which it is determined partly by considerations of torsion. For example consider an elliptic beam for which the plane of flexure passes through the minor axis. Let a_1 and b_1 be the semi-axes of the beam, and take its equation to be $x^2/b_1^2+y^2/a_1^2=1$ where $a_1>b_1$. Then we know that s is a maximum when $y=0$, $x=\pm b_1$, and its value at these points is $2\tau a_1^2 b_1/(a_1^2+b_1^2)$. Hence in this case the two maxima to be compared are

$$\beta_1(l-z)b_1, \quad \text{and} \quad \beta_1\sigma(l-z)b_1+\tau a_1^2 b_1/(a_1^2+b_1^2).$$

The greatest extension is therefore determined purely by flexure or partly by torsion according as

$$(l-z) > \text{or} < \frac{a_1^2}{a_1^2+b_1^2}\,\frac{\tau E a_1 b_1^3 \pi}{4X(1-\sigma)}.$$

Now each of the principal extensions has its maximum when $z=0$, and the extension at the highest point of the contour near the fixed end is the greatest extension anywhere, if the twisting couple G satisfy the inequality

$$G < \frac{\mu(1-\sigma)}{E}\,4Xl.$$

In this case the limit of safe loading is given by the formula (113) of the last article.

If G be greater than this limit, one of the principal extensions at the lowest point of the contour near the fixed end is the greatest, and is equal to one of the principal extensions at the highest point near the loaded end. In this case, according to the theory of Poncelet and Saint-Venant, the limit of safe loading is given by the equation

$$\Phi\left[\frac{E}{\mu}G + 4\sigma Xl\right] = \pi a b_1^2 T_0,$$

where T_0 is the breaking stress for pull in the direction of the axis of the beam, and Φ the factor of safety.

In the case of the ellipse, just worked out, it happens that s has its maximum where x has its maximum. When this is not the case, and it usually is not, the maximum value of $\beta_1\sigma lx + \frac{1}{2}s$ would require a special investigation. For example we might take the equilateral triangle; in general the lowest point of the contour is at a corner of the triangle, and s has its minimum value zero when x is a maximum. A case like that worked out for the ellipse would be fallen on by supposing the lowest side horizontal. Another example would be found by bending a horizontal elliptic bar with its major axis vertical by an applied weight, and twisting it by an applied couple. The maximum of x is then at the lowest point, which is the point where s is least. This case can be easily investigated and may be left to the reader.

111. Distortion of Cross-Section. Symmetrical Case.

We shall now suppose that the principal axes of inertia of the section coincide with axes of symmetry of the material of the beam, and that the two principal rigidities, corresponding to sections through these axes and the axis of the beam, are equal[1].

Then the problem of finding the terms of the general solution that contain β_1 reduces to finding ϕ to satisfy the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{.....(115),}$$

and the boundary-condition obtained from (93), viz.:

$$\frac{\partial\phi}{\partial n} = \beta_1\left[\tfrac{1}{2}l\sigma_1 x^2 + (E/\mu - \sigma_1)(mxy + \tfrac{1}{2}ly^2) - \sigma_2 ly^2\right] \text{...(116),}$$

[1] This supposition will be retained in the following articles, except where there is a statement modifying it.

where dn is the element of the normal to the bounding curve drawn outwards, and μ is written for M or L.

As in the case of torsion we may introduce a hydrodynamical analogy. For ϕ is the velocity potential of the plane motion of fluid within the bounding curve when the normal velocity at any point of the boundary has components

$$\beta_1 [\tfrac{1}{2}\sigma_1 x^2 + \tfrac{1}{2}(E/\mu - \sigma_1 - 2\sigma_2) y^2] \text{ parallel to } x,$$

and $$\beta_1 (E/\mu - \sigma_1) xy \text{ parallel to } y.$$

We may utilise this analogy to shew that ϕ is of the third order in x and y. For suppose x and y very small, then the normal velocity at the boundary is of the second order of small quantities, and the velocity at every point must be of the second order at least, and therefore the function ϕ whose differential coefficients are the components of velocity must be of the third order.

There exists a solution of the equation for ϕ subject to the boundary-condition. For we know from elementary hydrodynamics that there exists a solution of the equation with the normal rate of variation of ϕ given at the boundary provided

$$\int \frac{\partial \phi}{\partial n} ds = 0,$$

the integral being taken right round the boundary.

Now

$$\int \frac{\partial \phi}{\partial n} ds = \beta_1 \int [\tfrac{1}{2} l \sigma_1 x^2 + (E/\mu - \sigma_1)(mxy + \tfrac{1}{2} l y^2) - \sigma_2 l y^2] \, ds$$

$$= \beta_1 \frac{E}{\mu} \iint x \, dx \, dy$$

over the section, and this vanishes since the axis of z passes through the centre of inertia of the section.

In the general case where $L \neq M$ we may take $x'\sqrt{M} = x$, $y'\sqrt{L} = y$, the boundary is transformed to a curve in a new plane of x', y', and the boundary-condition (93) becomes

$$\frac{\sqrt{\left\{\left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial y'}\right)^2\right\}}}{\sqrt{\left\{\frac{1}{M}\left(\frac{\partial f}{\partial x'}\right)^2 + \frac{1}{L}\left(\frac{\partial f}{\partial y'}\right)^2\right\}}} \frac{\partial \phi}{\partial n'}$$

$$= \text{a given function of } x', y' = \psi(x', y'), \text{ say,}$$

where $f(x, y) = 0$ is the equation of the boundary, and dn' is the

element of the normal to the transformed boundary. The differential equation becomes

$$\frac{\partial^2\phi}{\partial x'^2}+\frac{\partial^2\phi}{\partial y'^2}=0,$$

and there will be a solution if

$$\int\psi(x',y')\left\{\frac{\left(\frac{\partial f}{\partial x'}\right)^2+\left(\frac{\partial f}{\partial y'}\right)^2}{\frac{1}{M}\left(\frac{\partial f}{\partial x'}\right)^2+\frac{1}{L}\left(\frac{\partial f}{\partial y'}\right)^2}\right\}^{-\frac{1}{2}}ds'=0,$$

where ds' is the element of arc of the new boundary. Remembering that $\psi(x', y')$ is linear in l, m we see that it becomes linear in l', m', (the direction cosines of dn'), and contains the reciprocal of the radical as a factor. This factor therefore disappears and the theorem reduces to that in the symmetrical case.

112. Distortion of the sections of a circular Cylinder.

To find ϕ for a circular cylinder we transform to polar coordinates

$$x=r\cos\theta,\qquad y=r\sin\theta,$$

so that the boundary-condition (116) becomes

$$\frac{\partial\phi}{\partial r}=r^2\beta_1\left[\tfrac{1}{2}\sigma_1\cos^3\theta+\left(\tfrac{3}{2}\frac{E}{\mu}-\tfrac{3}{2}\sigma_1-\sigma_2\right)\cos\theta\sin^2\theta\right]\text{ when } r=a$$

$$=a^2\beta_1\left[\left(\tfrac{3}{8}\frac{E}{\mu}-\tfrac{1}{4}\sigma_2\right)\cos\theta+\left(\tfrac{1}{2}\sigma_1+\tfrac{1}{4}\sigma_2-\tfrac{3}{8}\frac{E}{\mu}\right)\cos 3\theta\right]\quad\ldots(117).$$

Now any function ϕ which satisfies (115) within a circle whose centre is the origin can be expressed in the form

$$\sum_1^\infty r^n(A_n\cos n\theta+B_n\sin n\theta),$$

and to satisfy the boundary-condition we take this to contain only the terms in A_1 and A_3, thus

$$\phi=\beta_1\left[\left(\tfrac{3}{8}\frac{E}{\mu}-\tfrac{1}{4}\sigma_2\right)a^2r\cos\theta+\tfrac{1}{3}\left(\tfrac{1}{2}\sigma_1+\tfrac{1}{4}\sigma_2-\tfrac{3}{8}\frac{E}{\mu}\right)r^3\cos 3\theta\right]$$

$$=\beta_1\left[\left(\tfrac{3}{8}\frac{E}{\mu}-\tfrac{1}{4}\sigma_2\right)a^2x+\tfrac{1}{3}\left(\tfrac{1}{2}\sigma_1+\tfrac{1}{4}\sigma_2-\tfrac{3}{8}\frac{E}{\mu}\right)(x^3-3xy^2)\right]\quad\ldots(118).$$

It is easy to see that the integral

$$\iint\left(x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)dxdy$$

over the circle vanishes identically, and thence that the twisting couple (97) vanishes, and the resultant stress across any section reduces to a transverse force X parallel to the axis x given by the equation

$$X = \tfrac{1}{4}E\beta_1\pi a^4,$$

and a bending couple M' about an axis parallel to the axis y, which is given by

$$M' = -\tfrac{1}{4}E\beta_1\pi a^4 z.$$

Suppose an equal and opposite couple applied at the end $z = l$ and produced by forces distributed over the end, such that the force per unit area is parallel to the axis z and is

$$-E\beta_1 lx,$$

then we shall have the solution for bending by transverse force X by adding to the displacements (91), in which

$$\beta_1 = 4X/(E\pi a^4),$$

the displacements (29), in which

$$\alpha_1 = -4Xl/(E\pi a^4).$$

The displacements are, by (105) and (118),

$$\left.\begin{aligned} u &= \frac{2X}{E\pi a^4}\left[(l-z)(\sigma_1 x^2 - \sigma_2 y^2) + z^2\left(l - \tfrac{1}{3}z\right)\right] \\ v &= \frac{4X}{E\pi a^4}\left[(l-z)\,\sigma_2 xy\right] \\ w &= \frac{2X}{E\pi a^4}\left[\left(\tfrac{3}{4}\frac{E}{\mu} - \tfrac{1}{2}\sigma_2\right)a^2 x + \tfrac{1}{3}\left(\sigma_1 + \tfrac{1}{2}\sigma_2 - \tfrac{3}{4}\frac{E}{\mu}\right)(x^3 - 3xy^2)\right. \\ &\qquad \left. + (z^2 - 2lz)\,x + \left(\sigma_1 + \sigma_2 - \frac{E}{\mu}\right)xy^2\right] \end{aligned}\right\} \quad (119).$$

The change of shape of any rectangular element of the cross-section will be similar to that investigated in art. 91, so that lines $x = \text{const.}$ in the section remain approximately straight, and lines $y = \text{const.}$ become approximately arcs of circles whose concavity is in the opposite direction to that of the axis of the beam, and all the displacements in the plane of a section which affect the shape of an element are proportional to the distance from the end at which the force is applied.

But the point of most interest is the distortion of the sections into curved surfaces. The curves for which the displacement parallel to the axis is constant, called *curves of equal distortion*, are given by the equation

$$x(\alpha x^2 + \beta y^2 - 1) = \text{const.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (120),$$

where

$$\left.\begin{aligned} \alpha &= \frac{\frac{1}{3}\mu(\sigma_1+\frac{1}{2}\sigma_2)-\frac{1}{4}E}{\mu z(2l-z)-a^2(\frac{3}{4}E-\frac{1}{2}\mu\sigma_2)} \\ \beta &= \frac{\frac{1}{2}\mu\sigma_2-\frac{1}{4}E}{\mu z(2l-z)-a^2(\frac{3}{4}E-\frac{1}{2}\mu\sigma_2)} \end{aligned}\right\} \quad \text{.....(121).}$$

If αx^2 and βy^2 be small, these curves are very nearly straight lines $x = \text{const.}$; and thus, if l be great compared with a, near the end $z = l$, where the force X is applied, any section remains very approximately plane and is turned through such an angle as will make it pass through the centre of curvature of the arc into which the central-line is bent. As we travel along the beam from the loaded end towards the fixed end, z is continually diminishing, and at distances from the loaded end comparable with the radius of the section α and β are comparable with a^{-2}, so that the curves are no longer approximately straight.

113. Distortion of sections of isotropic circular beam by flexure.

Consider in particular the case of an isotropic beam for sections so near the fixed end that z may be neglected in comparison with a, and suppose (with Poisson) $\sigma_1 = \sigma_2 = \frac{1}{4}$, and $E = \frac{5}{2}\mu$, then the curves of equal distortion become

$$x(x^2 + y^2 - \tfrac{7}{2}a^2) = \text{const.} \quad \text{.....(122).}$$

In tracing the curves, if we take the constant zero, the curve consists of the axis y, and the circle $2x^2 + 2y^2 = 7a^2$, which completely surrounds the circular cross-section of the cylinder. The latter is the inner circle, and the former the outer circle in the figure. If we take the constant positive, and not too great, the curve consists of an oval lying within the outer circle on the side x positive, and an open branch outside this circle and asymptotic to the axis y on the side x negative. When the constant is $\frac{7}{3}\sqrt{\frac{7}{6}}$, the oval contracts to a point ($x = \sqrt{\frac{7}{6}}a$, $y = 0$) and for any greater value of the constant the oval disappears; this point is outside the circular boundary. If we take the constant negative similar results hold. The curves of equal distortion are the *parts* of the ovals included *within the inner circle*, and the displacement parallel to the z axis has the opposite sign to x. Thus if, to fix ideas, the axis of z be perpendicular to the plane of the paper, and the axis x vertically downwards, and the beam be supposed bent

by a vertical downwards force applied at the end $z = l$, all the part of the section in the lower half of the figure is shifted towards the origin, and all the part of the section in the upper half of the figure is shifted away from the origin, and the distance through which any point is shifted is the same at all points of the curves traced in fig. 16. (One quadrant only is drawn.)

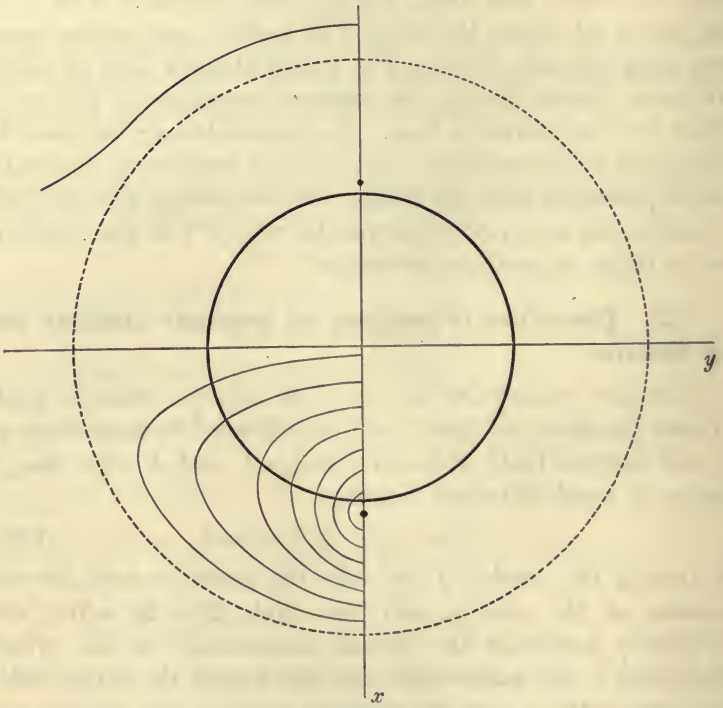

Fig. 16.

As we travel along the beam towards the end $z = l$ the outer circle expands. The double points move off farther from the inner circle and the parts of the ovals within the inner circle flatten out. If $l > \frac{1}{2}\sqrt{7}a$[1], the outer circle becomes infinite for $z = l - \sqrt{(l^2 - \frac{7}{4}a^2)}$, and, for greater values of z, α and β are negative, and the outer circle disappears. The curves of equal distortion may now be written

$$x(x^2 + y^2 + p) = q,$$

[1] See footnote p. 183.

where p is a constant depending on z, and q has the opposite sign to the displacement w. The curves of equal distortion are like that shown in fig. 17.

The surfaces into which the cross-sections are distorted are given by the equations

$$z - z_0 = \frac{2X}{E\pi a^4} \{z_0 (2l - z_0) - \tfrac{7}{4}a^2\} x (\alpha x^2 + \beta y^2 - 1),$$

where $z = z_0$ is the equation of any cross-section before strain. When z_0 is small this becomes

$$z - z_0 = -\frac{X}{E\pi a^4} x (x^2 + y^2 - \tfrac{7}{2}a^2),$$

of which the contour lines are drawn in fig. 16. When z_0 is greater than $l - \sqrt{(l^2 - \tfrac{7}{4}a^2)}$ the equation becomes of the form

$$z - z_0 = kx (x^2 + y^2 + p),$$

where k and p are constants. The contour-lines of this surface are such as that drawn in fig. 17.

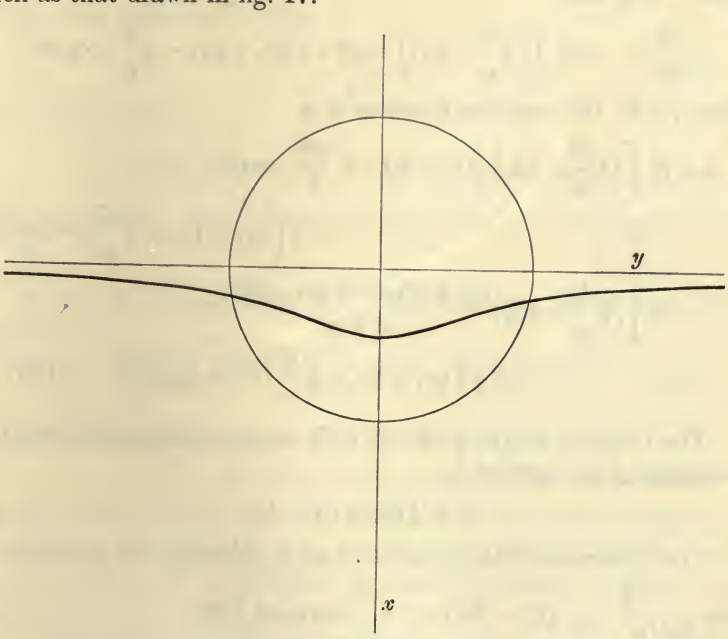

Fig. 17.

It is worth while to remark that the parts of the ovals in fig. 16 and of the curve in fig. 17 situated outside the circle $r = a$, are

the curves of equal distortion of the section of a hollow circular tube of internal radius a, and external radius r, bent to the same curvature by end tractions, and whose *outer* surface is subjected to traction parallel to the axis z. The amount of this traction near the fixed end is proportional to $x(a^2-r^2)/r$.

114. Distortion of sections of hollow circular beam.

Suppose the bounding curve of the normal section of the beam consists of two concentric circles, and let $r=a$ be the outer boundary, and $r=b$ the inner. Then we have to find a function ϕ to satisfy the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

at all points between the two circles, and

$$\frac{\partial\phi}{\partial r}=a^2\beta_1\left[\left(\tfrac{3}{8}\frac{E}{\mu}-\tfrac{1}{4}\sigma_2\right)\cos\theta+\left(\tfrac{1}{2}\sigma_1+\tfrac{1}{4}\sigma_2-\tfrac{3}{8}\frac{E}{\mu}\right)\cos 3\theta\right]$$

when $r=a$, and

$$-\frac{\partial\phi}{\partial r}=-b^2\beta_1\left[\left(\tfrac{3}{8}\frac{E}{\mu}-\tfrac{1}{4}\sigma_2\right)\cos\theta+\left(\tfrac{1}{2}\sigma_1+\tfrac{1}{4}\sigma_2-\tfrac{3}{8}\frac{E}{\mu}\right)\cos 3\theta\right]$$

when $r=b$; the required function ϕ is

$$\begin{aligned}\phi&=\beta_1\left[\left(\tfrac{3}{8}\frac{E}{\mu}-\tfrac{1}{4}\sigma_2\right)\left\{(a^2+b^2)\,r+\frac{a^2b^2}{r}\right\}\cos\theta\right.\\&\qquad\left.+\tfrac{1}{3}\left(\tfrac{1}{2}\sigma_1+\tfrac{1}{4}\sigma_2-\tfrac{3}{8}\frac{E}{\mu}\right)r^3\cos 3\theta\right]\\&=\beta_1\left[\left(\tfrac{3}{8}\frac{E}{\mu}-\tfrac{1}{4}\sigma_2\right)\frac{(a^2+b^2)(x^2+y^2)+a^2b^2}{x^2+y^2}x\right.\\&\qquad\left.+\tfrac{1}{3}\left(\tfrac{1}{2}\sigma_1+\tfrac{1}{4}\sigma_2-\tfrac{3}{8}\frac{E}{\mu}\right)(x^3-3xy^2)\right]\ldots\ldots(123).\end{aligned}$$

The twisting couple given by (97) vanishes identically, and the transverse force applied is

$$X=\tfrac{1}{4}E\beta_1\pi(a^4-b^4).$$

The distortion of the cross-sections is given by the equation

$$\begin{aligned}w=\frac{2X}{E\pi(a^4-b^4)}&\left[(z^2-2lz)\,x-\left(\frac{E}{\mu}-\sigma_1-\sigma_2\right)xy^2\right.\\&\left.+\left(\tfrac{3}{4}\frac{E}{\mu}-\tfrac{1}{2}\sigma_2\right)\frac{(a^2+b^2)(x^2+y^2)+a^2b^2}{x^2+y^2}x+\tfrac{1}{3}\left(\sigma_1+\tfrac{1}{2}\sigma_2-\tfrac{3}{4}\frac{E}{\mu}\right)(x^3-3xy^2)\right]\\&\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots(124),\end{aligned}$$

and the equations of the curves of equal distortion are of the form

$$x\left(1-\alpha x^2-\beta y^2-\frac{\gamma}{x^2+y^2}\right)=\text{const.},$$

where all the constants that occur are functions of z.

For an isotropic solid fulfilling Poisson's condition ($\sigma=\frac{1}{4}$), it can be easily verified that these curves, when z is small enough to be neglected, are given by the equation

$$x\left[(x^2+y^2)^2-\tfrac{7}{2}(a^2+b^2)(x^2+y^2)-\tfrac{7}{2}a^2b^2\right]=-4w(x^2+y^2)/\beta_1 \ldots (125),$$

and they may be traced in the same manner as the curves for a solid cylinder.

115. Elliptic cylinder.

To simplify this case we shall suppose the beam isotropic. We have then to find a function ϕ to satisfy the differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

at all points within the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (126),$$

and the condition

$$\frac{\partial\phi}{\partial n}=\beta_1\left[\tfrac{1}{2}l\left\{\sigma x^2+(2-\sigma)y^2\right\}+m(2+\sigma)xy\right]$$

at the boundary. This is obtained from (116) by writing $2(1+\sigma)$ for E/μ.

If p be the perpendicular from the centre on any tangent, this condition is

$$\frac{\partial\phi}{\partial n}=\beta_1 p\left[\frac{\sigma x^3+(2-\sigma)y^2x}{2a^2}+\frac{(2+\sigma)y^2x}{b^2}\right] \ldots\ldots (127).$$

Take conjugate functions ξ, η of x, y such that

$$x+\iota y=c\cosh(\xi+\iota\eta) \ldots\ldots\ldots\ldots\ldots\ldots (128),$$

where $c^2=a^2-b^2$, then we know that, if ϕ be the real or imaginary part of any function of $\xi+\iota\eta$, the equation will be satisfied and the form of ϕ will be adapted to satisfy boundary-conditions at the surface of the elliptic cylinder.

Let the cylinder be the surface $\xi=\xi_0$ of the family, so that

$$c\cosh\xi_0=a,$$
$$c\sinh\xi_0=b,$$

and let p be the perpendicular from the axis on the tangent plane at any point, and h the positive value of the function

$$\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2\right]^{\frac{1}{2}}.$$

Then we know that $h^{-1}d\xi$ is the distance between the consecutive surfaces ξ and $\xi+d\xi$, and $h^{-1}d\eta$ is the distance between consecutives η and $\eta+d\eta$; also we have

$$h^{-2}=\tfrac{1}{2}c^2(\cosh 2\xi-\cos 2\eta),$$

from which, by putting $\xi=\xi_0$, we find for the perpendicular p from the centre of the elliptic section on any tangent the equation

$$hab=p \qquad \text{.....(129).}$$

The solutions ϕ of the differential equation, which remain finite continuous and one-valued within the elliptic boundary, are of the form

$$\phi=\beta_1\Sigma\,[A_n\cosh n\xi\cos n\eta+B_n\sinh n\xi\sin n\eta] \qquad \text{...(130).}$$

The boundary-condition (127) at $\xi=\xi_0$ is

$$h\frac{\partial\phi}{\partial\xi}=\beta_1 p\left[\left\{(2+\sigma)\frac{c^3}{b^2}+(1-\tfrac{1}{2}\sigma)\frac{c^3}{a^2}\right\}\cosh\xi_0\sinh^2\xi_0\cos\eta\sin^2\eta+\tfrac{1}{2}\sigma\frac{c^3}{a^2}\cosh^3\xi_0\cos^3\eta\right],$$

or

$$\frac{\partial\phi}{\partial\xi}=\beta_1 ab\left[\left\{(2+\sigma)a+(1-\tfrac{1}{2}\sigma)\frac{b^2}{a}\right\}\tfrac{1}{4}(\cos\eta-\cos 3\eta)+\tfrac{1}{8}\sigma a(\cos 3\eta+3\cos\eta)\right] \qquad \text{.....(131).}$$

Hence all the B's vanish; and, of the A's, all vanish except A_1 and A_3, and we have

$$\left.\begin{aligned} A_1\sinh\xi_0&=\tfrac{1}{4}ab\left[(2+\sigma)a+(1-\tfrac{1}{2}\sigma)\frac{b^2}{a}+\tfrac{3}{2}\sigma a\right]\\ 3A_3\sinh 3\xi_0&=-\tfrac{1}{4}ab\left[(2+\sigma)a+(1-\tfrac{1}{2}\sigma)\frac{b^2}{a}-\tfrac{1}{2}\sigma a\right]\end{aligned}\right\},$$

or

$$A_1=\tfrac{1}{4}ac\left[(2+\tfrac{5}{2}\sigma)a+(1-\tfrac{1}{2}\sigma)\frac{b^2}{a}\right],$$

and

$$A_3=-\tfrac{1}{12}\frac{abc^3}{3c^2b+4b^3}\left[(2+\tfrac{1}{2}\sigma)a+(1-\tfrac{1}{2}\sigma)\frac{b^2}{a}\right].$$

Hence

$$\phi = \tfrac{1}{4}\beta_1 ac\left[(2+\tfrac{5}{2}\sigma)\,a + (1-\tfrac{1}{2}\sigma)\frac{b^2}{a}\right]\cosh\xi\cos\eta$$

$$-\tfrac{1}{12}\beta_1\frac{ac^3}{3c^2+4b^2}\left[(2+\tfrac{1}{2}\sigma)\,a + (1-\tfrac{1}{2}\sigma)\frac{b^2}{a}\right]\cosh 3\xi\cos 3\eta \ldots (132).$$

Now

$$\cosh 3\xi\cos 3\eta = \frac{x}{c}\left[\frac{4x^2-12y^2-3c^2}{c^2}\right],$$

as is easily verified; hence

$$\phi = \tfrac{1}{4}\beta_1 x\left[(2+\tfrac{5}{2}\sigma)\,a^2 + (1-\tfrac{1}{2}\sigma)\,b^2\right]$$

$$-\tfrac{1}{12}\beta_1 x\frac{4x^2-12y^2-3c^2}{3c^2+4b^2}\left[(2+\tfrac{1}{2}\sigma)\,a^2 + (1-\tfrac{1}{2}\sigma)\,b^2\right],$$

or

$$\phi = \frac{\beta_1 x}{3a^2+b^2}\left[a^2\{2a^2+b^2+2\sigma a^2\} - \tfrac{1}{3}(x^2-3y^2)\{2a^2+b^2+\tfrac{1}{2}\sigma(a^2-b^2)\}\right]$$

.....(133).

We notice by way of verification that the value of ϕ for the circle is deduced by putting $a = b$.

It is easy to see that the couple (97) about the axis z vanishes.

As in the case of the circle we find the displacements produced by a load X at the end $z = l$, by substituting from (133) in (105), in the forms:

$$\left.\begin{aligned} u &= \frac{2X}{E\pi a^3 b}\left[\sigma(l-z)(x^2-y^2) + z^2(l-\tfrac{1}{3}z)\right] \\ v &= \frac{4X}{E\pi a^3 b}\,\sigma(l-z)\,xy \\ w &= \frac{2X}{E\pi a^3 b}\,x\left[(2l-z)\,z - \frac{2a^2(2a^2+b^2+2\sigma a^2)}{3a^2+b^2}\right]\left[\alpha x^2 + \beta y^2 - 1\right] \end{aligned}\right\}$$

.....(134),

where

$$\left.\begin{aligned} \alpha &= \frac{\tfrac{1}{3}\{4a^2+2b^2+\sigma(a^2-b^2)\}}{-(3a^2+b^2)(2l-z)\,z + 2a^2(2a^2+b^2+2\sigma a^2)} \\ \beta &= \frac{2a^2-\sigma(a^2-b^2)}{-(3a^2+b^2)(2l-z)\,z + 2a^2(2a^2+b^2+2\sigma a^2)} \end{aligned}\right\}$$

....(135),

and the curves of equal distortion are now ovals lying within an ellipse when z is small, and curves like that in fig. 17 when z is

considerable. These are among the curves given in the frontispiece of Prof. Pearson's *Elastical Researches of Barré de Saint-Venant.*

116. Flexure of Æolotropic Rectangular Beam.

We shall suppose the sides of the rectangle parallel to the axes of x and y, and of lengths $2a$ and $2b$.

When the two principal rigidities of the beam in the planes (z, x) and (z, y) are unequal we have to satisfy the equation

$$M\frac{\partial^2\phi}{\partial x^2}+L\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{.....(136),}$$

within the rectangle, and the boundary-conditions

$$\frac{\partial\phi}{\partial x}=\tfrac{1}{2}\beta_1\left[\sigma_1 x^2+\frac{E-M\sigma_1-2L\sigma_2}{L}y^2\right] \quad \text{.....(137),}$$

when $x=\pm a$ and $b>y>-b$,

and

$$\frac{\partial\phi}{\partial y}=\beta_1\frac{E-M\sigma_1}{L}xy \quad \text{.....(138),}$$

when $y=\pm b$ and $a>x>-a$.

Now, as in the case of torsion, take

$$\left.\begin{aligned} x' &= x\sqrt{(1+L/M)} \\ y' &= y\sqrt{(1+M/L)} \end{aligned}\right\} \quad \text{.....(139).}$$

Then the differential equation becomes

$$\frac{\partial^2\phi}{\partial x'^2}+\frac{\partial^2\phi}{\partial y'^2}=0,$$

and the boundary-conditions become

$$\frac{\partial\phi}{\partial x'}=\tfrac{1}{2}\beta_1\sqrt{\left(\frac{M}{L+M}\right)}\left[\sigma_1\frac{M}{L+M}x'^2+\frac{E-M\sigma_1-2L\sigma_2}{L+M}y'^2\right],$$

when $x'=\pm a\sqrt{(1+L/M)}$

and $b\sqrt{(1+M/L)}>y'>-b\sqrt{(1+M/L)}$,

and

$$\frac{\partial\phi}{\partial y'}=\beta_1\sqrt{\left(\frac{M}{L+M}\right)}\frac{E-M\sigma_1}{L+M}x'y',$$

when $y'=\pm b\sqrt{(1+M/L)}$

and $a\sqrt{(1+L/M)}>x'>-a\sqrt{(1+L/M)}$.

Take a new function χ such that

$$\phi=-\tfrac{1}{6}\beta_1\sqrt{\left(\frac{M}{L+M}\right)}\frac{E-M\sigma_1}{L+M}(x'^3-3x'y'^2)+\chi \quad \text{.....(140).}$$

Then χ must be determined to satisfy the equation

$$\frac{\partial^2\chi}{\partial x'^2}+\frac{\partial^2\chi}{\partial y'^2}=0$$

at all points of the section, and the boundary-conditions

$$\frac{\partial\chi}{\partial x'}=\beta_1\sqrt{\left(\frac{M}{L+M}\right)}\left[\frac{E}{2(L+M)}x'^2-\frac{L\sigma_2}{L+M}y'^2\right]$$

at the x' boundary, and

$$\frac{\partial\chi}{\partial y'}=0$$

at the y' boundary.

Assume

$$\chi=\beta_1\sqrt{\left(\frac{M}{L+M}\right)}\left[Ax'+\sum_1^\infty\left(A_n\sinh\frac{n\pi x'}{b'}\cos\frac{n\pi y'}{b'}\right)\right]\ldots.(141),$$

where a' is written for $a\sqrt{(1+L/M)}$, and b' for $b\sqrt{(1+M/L)}$. This form is taken because $\dfrac{\partial\chi}{\partial x'}$ is an even function, and the condition at the y' boundary is satisfied identically. To determine the coefficients we have

$$A+\sum_1^\infty\left[\frac{n\pi}{b'}A_n\cosh\frac{n\pi a'}{b'}\cos\frac{n\pi y'}{b'}\right]=\frac{E}{2M}a^2-\frac{L\sigma_2}{L+M}y'^2,$$

when $b'>y'>-b'$. Now between these limits we may expand y'^2 by Fourier's theorem in the form

$$y'^2=\frac{b'^2}{3}+\frac{4b'^2}{\pi^2}\sum_1^\infty\left[\frac{(-)^n}{n^2}\cos\frac{n\pi y'}{b'}\right].$$

Hence

$$A=\frac{E}{2M}a^2-\sigma_2\frac{b^2}{3},$$

$$A_n=-\sqrt{\left(\frac{L+M}{L}\right)}\frac{4b^3}{\pi^3}\sigma_2\frac{(-)^n}{n^3}\operatorname{sech}\left\{\frac{n\pi a}{b}\sqrt{\left(\frac{L}{M}\right)}\right\}.$$

And therefore

$$\phi=\beta_1\left\{\left(\frac{E}{2M}a^2-\tfrac{1}{3}\sigma_2b^2\right)x-\tfrac{1}{6}(E-M\sigma_1)\left(\frac{x^3}{M}-\frac{3xy^2}{L}\right)\right.$$

$$\left.-\frac{4b^3}{\pi^3}\sqrt{\frac{M}{L}}\sigma_2\sum_1^\infty\left[\frac{(-)^n}{n^3}\frac{\sinh\left\{\dfrac{n\pi x}{b}\sqrt{\left(\dfrac{L}{M}\right)}\right\}}{\cosh\left\{\dfrac{n\pi a}{b}\sqrt{\left(\dfrac{L}{M}\right)}\right\}}\cos\frac{n\pi y}{b}\right]\right\}\ldots(142).$$

For an isotropic beam this reduces to

$$\phi = \beta_1 \left\{ [(1+\sigma)\, a^2 - \tfrac{1}{3}\sigma b^2]\, x - \tfrac{1}{6}(2+\sigma)(x^3 - 3xy^2) \right.$$
$$\left. - \frac{4b^3}{\pi^3}\sigma \sum_1^\infty \left[\frac{(-)^n}{n^3} \frac{\sinh \frac{n\pi x}{b}}{\cosh \frac{n\pi a}{b}} \cos \frac{n\pi y}{b} \right] \right\} \quad \ldots\ldots(143).$$

Returning to the general case, it is easy to see by symmetry that the integrals $\iint x \frac{\partial \phi}{\partial y}\, dxdy$, and $\iint y \frac{\partial \phi}{\partial x}\, dxdy$ vanish, and that therefore the terms in β_1 contribute no couple about the axis z.

Now suppose the beam bent by a load X parallel to x applied at the end $z = l$, we have, as in art. 107, to add the solutions for

$$\beta_1 = \tfrac{3}{4}X/(Ea^3b),$$
$$\alpha_1 = -\tfrac{3}{4}Xl/(Ea^3b).$$

The displacements are

$$\left.\begin{aligned} u &= \tfrac{3}{8}\frac{X}{Ea^3b}\left[(l-z)(\sigma_1 x^2 - \sigma_2 y^2) - (l - \tfrac{1}{3}z)\, z^2\right], \\ v &= \tfrac{3}{4}\frac{X}{Ea^3b}\sigma_2 (l-z)\, xy, \\ w &= \tfrac{3}{8}\frac{X}{Ea^3b}\left[z(2l-z)\, x - (E - M\sigma_1 - L\sigma_2)\, xy^2/L\right] + \phi \end{aligned}\right\} (144),$$

where ϕ is the function determined by (142).

The curves of equal distortion have been traced by Saint-Venant for a square beam of isotropic matter obeying Poisson's condition.

Investigations by means of conjugate functions might be given of the distortion of beams of various forms of section by flexure, but the problem is less interesting than the corresponding one of torsion on account of the comparative smallness of ϕ.

CHAPTER VII.

CURVILINEAR COORDINATES.

117. Orthogonal Surfaces.

For many problems it is convenient to use systems of curvilinear coordinates instead of the ordinary Cartesian rectangular coordinates. These may be defined as follows:

Let $f(x, y, z) = \alpha$, some constant, be the equation of a family of surfaces. Fixing our attention upon any point (x, y, z), one surface of the family will in general pass through this point. If small variations be made in x, y, z, *i.e.* if we pass to a neighbouring point $(x + dx, y + dy, z + dz)$, this point will in general lie on a surface of the family differing from the surface $f(x, y, z) = \alpha$, but near to it. The surface on which it lies is given by the equation $f(x, y, z) = \alpha + d\alpha$, where

$$d\alpha = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \frac{\partial \alpha}{\partial x}dx + \frac{\partial \alpha}{\partial y}dy + \frac{\partial \alpha}{\partial z}dz.$$

Thus a knowledge of α gives the surface of the family on which the point (x, y, z) lies, and α is called a curvilinear coordinate of the point (x, y, z).

If now we take three independent families of surfaces

$$f_1(x, y, z) = \alpha,$$
$$f_2(x, y, z) = \beta,$$
$$f_3(x, y, z) = \gamma,$$

and fix our attention on the point (x, y, z), we find one surface of each family passing through the point. If a neighbouring point be taken one surface of each family will pass through the neighbouring point. The two sets of surfaces are taken to be (α, β, γ) for the

point (x, y, z), and $(\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$ for the neighbouring point. The quantities (α, β, γ) are called curvilinear coordinates of the point. Now, conversely, as any point will lie on three particular surfaces these determine the point; and, the region of space considered being suitably limited, if we attach to one point of this region a set of corresponding values of (α, β, γ), and proceed in all directions from this point, by giving to (x, y, z) as functions of (α, β, γ) values continuous with those at the chosen starting point, any point within the region will be given by its (α, β, γ).

The most convenient systems to choose, in applications of the theory of elasticity, are systems of surfaces which cut each other everywhere at right angles. Such systems are called orthogonal surfaces. It is well known that there exists an infinite number of sets of such surfaces, and, according to a celebrated theorem of Dupin's, the intersection of two surfaces belonging to different families of the same set of orthogonal surfaces is a line of curvature on each. In what follows we shall suppose the surfaces to be α, β, γ, and shall suppose that these cut each other everywhere at right angles, so that the three relations

$$\left.\begin{aligned} \frac{\partial\beta}{\partial x}\frac{\partial\gamma}{\partial x} + \frac{\partial\beta}{\partial y}\frac{\partial\gamma}{\partial y} + \frac{\partial\beta}{\partial z}\frac{\partial\gamma}{\partial z} &= 0, \\ \frac{\partial\gamma}{\partial x}\frac{\partial\alpha}{\partial x} + \frac{\partial\gamma}{\partial y}\frac{\partial\alpha}{\partial y} + \frac{\partial\gamma}{\partial z}\frac{\partial\alpha}{\partial z} &= 0, \\ \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y} + \frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial z} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

are identically satisfied.

The theory of orthogonal curvilinear coordinates is due to Lamé, and was developed by him in his *Léçons sur les coordonnées curvilignes*. The method we shall employ is founded on the particular case treated by Mr Webb in the *Messenger of Mathematics*, 1882. The problems at the end of the chapter have been considered by various writers, including Poisson, Lamé, Clebsch, Saint-Venant, and Mr Chree.

118. The line-element.

Let dn_1 be the length cut off from the normal to $\alpha =$ constant at any point (x, y, z) by the neighbouring surface $\alpha + d\alpha$ of the family, and write h_1^2 for the quantity

$$(\partial\alpha/\partial x)^2 + (\partial\alpha/\partial y)^2 + (\partial\alpha/\partial z)^2.$$

Then, if $x+dx$, $y+dy$, $z+dz$ be the coordinates of the point in which the normal to α at (x, y, z) meets the surface $\alpha+d\alpha$, we shall have, by projecting the line joining two neighbouring points on the normal to α,

$$dn_1 = \frac{1}{h_1}\left(\frac{\partial\alpha}{\partial x}dx + \frac{\partial\alpha}{\partial y}dy + \frac{\partial\alpha}{\partial z}dz\right) = d\alpha/h_1.$$

If, in like manner, dn_2, dn_3 be the elements of the normals to β and γ, drawn through the point (x, y, z), we shall find

$$dn_2 = d\beta/h_2, \qquad dn_3 = d\gamma/h_3,$$

where $$h_2^2 = (\partial\beta/\partial x)^2 + (\partial\beta/\partial y)^2 + (\partial\beta/\partial z)^2,$$

and $$h_3^2 = (\partial\gamma/\partial x)^2 + (\partial\gamma/\partial y)^2 + (\partial\gamma/\partial z)^2.$$

Since the square of the distance between the points (α, β, γ), $(\alpha+d\alpha, \beta+d\beta, \gamma+d\gamma)$ is $dn_1^2+dn_2^2+dn_3^2$, we find for this distance the expression

$$[(d\alpha/h_1)^2 + (d\beta/h_2)^2 + (d\gamma/h_3)^2]^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In general h_1, h_2, h_3 can be supposed to be expressed in terms of α, β, γ. The quantity (2) is called the line-element.

119. Vector-Differentiation.

If P be a point whose coordinates are x, y, z, we may draw through P a system of rectangular axes, to which we may refer points in the neighbourhood of P, the directions of the axes being the normals to the three surfaces α, β, γ which pass through P. If x_1, y_1, z_1 represent the coordinates of any point near to P, referred to this system of axes at P, we require formulæ for differentiation with respect to x_1, y_1, z_1. It is plain that dx_1, dy_1, dz_1 are the same as dn_1, dn_2, dn_3, but it does not follow that $\dfrac{\partial\phi}{\partial x_1}$ is the same as $h_1\dfrac{\partial\phi}{\partial\alpha}$, when ϕ is a component of a vector, which has a magnitude and direction depending on (α, β, γ). In estimating $\dfrac{\partial\phi}{\partial x_1}$, we have to remember that the change contemplated in ϕ is that which takes place when we pass from P to a near point situated on the normal to α at P. If ϕ be a component of a vector quantity estimated parallel to one or other of the three normals at any point, the change produced in ϕ, when we pass from any point to a neighbouring point, will depend partly on the change of direction of the axis along which the vector is resolved.

Now let ξ, η, ζ be the components of a vector quantity parallel to the three normals at any point (α, β, γ), then we know that the changes, $\delta\xi$, $\delta\eta$, $\delta\zeta$, which take place when we pass from any point (α, β, γ) to a neighbouring point $(\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$ are

$$\left.\begin{aligned} \delta\xi &= d\xi - \eta\delta\theta_3 + \zeta\delta\theta_2, \\ \delta\eta &= d\eta - \zeta\delta\theta_1 + \xi\delta\theta_3, \\ \delta\zeta &= d\zeta - \xi\delta\theta_2 + \eta\delta\theta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

in which $d\xi$ stands for $\dfrac{\partial\xi}{\partial\alpha}d\alpha + \dfrac{\partial\xi}{\partial\beta}d\beta + \dfrac{\partial\xi}{\partial\gamma}d\gamma$, and $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ are the infinitesimal rotations of the three normals at (α, β, γ) about themselves necessary to bring them into coincidence with the normals at $(\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$.

120. The three rotations.

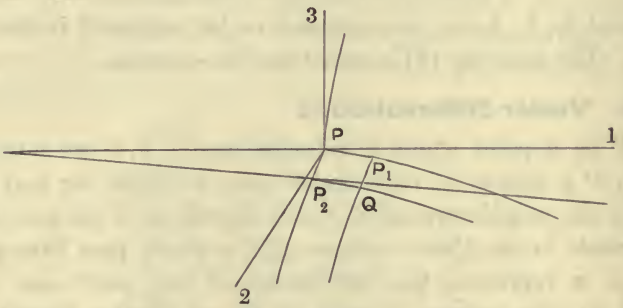

Fig. 18.

Let P be the point (α, β, γ); P_1, $(\alpha + d\alpha, \beta, \gamma)$; P_2, $(\alpha, \beta + d\beta, \gamma)$; and Q, $(\alpha + d\alpha, \beta + d\beta, \gamma)$.

It is clear that the rotation $\delta\theta_3$ will be a linear function of $d\alpha$, $d\beta$. To find the term in $d\beta$, observe that the length PP_2 is $\dfrac{d\beta}{h_2}$, and the length P_1Q is $\dfrac{d\beta}{h_2} + d\alpha\dfrac{\partial}{\partial\alpha}\left(\dfrac{d\beta}{h_2}\right)$; the angle between the tangents to PP_1 and P_2Q is found by dividing the difference of these by PP_1, and is therefore equal to

$$h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)d\beta,$$

and this is the rotation of the system of axes from (1) towards

(2) in passing from P to P_2, *i.e.* it is the term of $\delta\theta_3$ that contains $d\beta$. In like manner the term in $d\alpha$ is

$$-h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)d\alpha.$$

Adding these we get the complete expression for $\delta\theta_3$. We can now put down the formulæ

$$\left.\begin{aligned}\delta\theta_1 &= h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)d\gamma - h_3\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)d\beta,\\ \delta\theta_2 &= h_3\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)d\alpha - h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)d\gamma,\\ \delta\theta_3 &= h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)d\beta - h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)d\alpha\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(4),$$

of which the third has just been proved, and the other two are found by cyclical interchanges of the letters and suffixes.

As an example of the application of these formulae we shall find the normal to the surface γ at $(\alpha+d\alpha,\ \beta,\ \gamma)$.

In the expressions (4) for $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ we must put $d\beta=0$ and $d\gamma=0$.

The equations giving the changes in the direction-cosines $(l,\ m,\ n)$ of any line are obtained from (3) by putting $l,\ m,\ n$ for $\xi,\ \eta,\ \zeta$. Thus

$$\begin{aligned}\delta l &= dl - m\delta\theta_3 + n\delta\theta_2,\\ \delta m &= dm - n\delta\theta_1 + l\delta\theta_3,\\ \delta n &= dn - l\delta\theta_2 + m\delta\theta_1.\end{aligned}$$

When $l=0,\ m=0,\ n=1$, these become

$$\begin{aligned}\delta l &= \delta\theta_2 = h_3\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)d\alpha,\\ \delta m &= 0,\\ \delta n &= 0;\end{aligned}$$

so that the equations of the normal to γ at $(\alpha+d\alpha,\ \beta,\ \gamma)$, referred to the three normals at $(\alpha,\ \beta,\ \gamma)$, are

$$y_1=0,\quad \frac{x_1-\dfrac{d\alpha}{h_1}}{h_3\dfrac{\partial}{\partial\gamma}\left(\dfrac{1}{h_1}\right)d\alpha}=\frac{z_1}{1}.$$

This meets the normal to γ at $(\alpha,\ \beta,\ \gamma)$ in the point

$$x_1=0,\ y_1=0,\ z_1=-\frac{1}{h_1h_3\dfrac{\partial}{\partial\gamma}\left(\dfrac{1}{h_1}\right)},$$

and thus the line of intersection of $\beta = \text{const.}$ and $\gamma = \text{const.}$ is a line of curvature on γ. This proves Dupin's Theorem, and gives for the principal curvature $1/\rho_1$ of the surface γ in the normal section through the line $d\alpha$

$$\frac{1}{\rho_1} = -h_1 h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_1}\right) \quad \ldots\ldots(5).$$

In like manner we could find the other principal curvature $1/\rho_2$ of γ, viz.:

$$\frac{1}{\rho_2} = h_2 h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_2}\right) \quad \ldots\ldots(6).$$

These formulæ for the curvatures are due to Lamé.

121. The strains.

We shall now find expressions for the components of strain and the rotations of an elastic medium referred to the orthogonal coordinates.

Suppose a system of rectangular axes drawn through any point $P(\alpha, \beta, \gamma)$ whose directions coincide with the normals to the three surfaces at the point. Let x_1, y_1, z_1 be the coordinates of a neighbouring point Q referred to this system of axes. Then after strain we must suppose the whole of the elastic body moved back without rotation so that P is brought to its old position. When this is done, let $x_1 + u$, $y_1 + v$, $z_1 + w$ be the coordinates of Q referred to the same system of axes. The six components of strain are the three extensions

$$\frac{\partial u}{\partial x_1}, \quad \frac{\partial v}{\partial y_1}, \quad \frac{\partial w}{\partial z_1},$$

and the three shears

$$\frac{\partial w}{\partial y_1} + \frac{\partial v}{\partial z_1}, \quad \frac{\partial u}{\partial z_1} + \frac{\partial w}{\partial x_1}, \quad \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial y_1},$$

and the rotations of the medium are

$$\tfrac{1}{2}\left(\frac{\partial w}{\partial y_1} - \frac{\partial v}{\partial z_1}\right), \quad \tfrac{1}{2}\left(\frac{\partial u}{\partial z_1} - \frac{\partial w}{\partial x_1}\right), \quad \tfrac{1}{2}\left(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial y_1}\right).$$

Since u, v, w are components of a vector, the changes in them are by (3)

$$\left.\begin{aligned} \delta u &= du - v\delta\theta_3 + w\delta\theta_2 \\ \delta v &= dv - w\delta\theta_1 + u\delta\theta_3 \\ \delta w &= dw - u\delta\theta_2 + v\delta\theta_1 \end{aligned}\right\} \quad \ldots\ldots(7).$$

Inserting the values of the $\delta\theta$'s from (4), the first of these equations becomes

$$\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial y_1}dy_1+\frac{\partial u}{\partial z_1}dz_1=\frac{\partial u}{\partial \alpha}d\alpha+\frac{\partial u}{\partial \beta}d\beta+\frac{\partial u}{\partial \gamma}d\gamma$$
$$-v\left[h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)d\beta-h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)d\alpha\right]+w\left[h_3\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)d\alpha-h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)d\gamma\right] \quad \ldots\ldots\ldots\ldots(8).$$

Remembering that dx_1, dy_1, dz_1 are $d\alpha/h_1$, $d\beta/h_2$, $d\gamma/h_3$ we obtain by equating the coefficients of $d\alpha$, $d\beta$, $d\gamma$ the results

$$\left.\begin{aligned}\frac{\partial u}{\partial x_1}&=h_1\frac{\partial u}{\partial\alpha}+h_1h_2v\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+h_1h_3w\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\\ \frac{\partial u}{\partial y_1}&=h_2\frac{\partial u}{\partial\beta}-h_1h_2v\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\\ \frac{\partial u}{\partial z_1}&=h_3\frac{\partial u}{\partial\gamma}-h_1h_3w\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)\end{aligned}\right\}\ldots\ldots\ldots(9).$$

Similar results follow from the other equations of (7).

If as in ch. I. we write e, f, g, a, b, c for the six components of strain

$$\frac{\partial u}{\partial x_1},\ \frac{\partial v}{\partial y_1},\ldots\ \frac{\partial v}{\partial x_1}+\frac{\partial u}{\partial y_1},$$

we find

$$\left.\begin{aligned}e&=h_1\frac{\partial u}{\partial\alpha}+h_1h_2v\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+h_1h_3w\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\\ f&=h_2\frac{\partial v}{\partial\beta}+h_2h_3w\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)+h_2h_1u\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\\ g&=h_3\frac{\partial w}{\partial\gamma}+h_3h_1u\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)+h_3h_2v\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)\\ a&=h_2\frac{\partial w}{\partial\beta}+h_3\frac{\partial v}{\partial\gamma}-h_2h_3\left[v\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)+w\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)\right]\\ b&=h_3\frac{\partial u}{\partial\gamma}+h_1\frac{\partial w}{\partial\alpha}-h_3h_1\left[w\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)+u\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\right]\\ c&=h_1\frac{\partial v}{\partial\alpha}+h_2\frac{\partial u}{\partial\beta}-h_1h_2\left[u\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)+v\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right]\end{aligned}\right\}\ldots\ldots(10).$$

These give the six strain-components, and the cubical dilatation Δ is given by the equation

$$\Delta=h_1h_2h_3\left[\frac{\partial}{\partial\alpha}\left(\frac{u}{h_2h_3}\right)+\frac{\partial}{\partial\beta}\left(\frac{v}{h_3h_1}\right)+\frac{\partial}{\partial\gamma}\left(\frac{w}{h_1h_2}\right)\right]\ldots\ldots(11).$$

Again, if as in ch. I. we write ϖ_1, ϖ_2, ϖ_3 for the three rotations

$$\tfrac{1}{2}\left(\frac{\partial w}{\partial y_1}-\frac{\partial v}{\partial z_1}\right),\ \tfrac{1}{2}\left(\frac{\partial u}{\partial z_1}-\frac{\partial w}{\partial x_1}\right),\ \tfrac{1}{2}\left(\frac{\partial v}{\partial x_1}-\frac{\partial u}{\partial y_1}\right),$$

we easily find the formulæ

$$\left.\begin{aligned} 2\varpi_1 &= h_2h_3\left[\frac{\partial}{\partial\beta}\left(\frac{w}{h_3}\right)-\frac{\partial}{\partial\gamma}\left(\frac{v}{h_2}\right)\right]\\ 2\varpi_2 &= h_3h_1\left[\frac{\partial}{\partial\gamma}\left(\frac{u}{h_1}\right)-\frac{\partial}{\partial\alpha}\left(\frac{w}{h_3}\right)\right]\\ 2\varpi_3 &= h_1h_2\left[\frac{\partial}{\partial\alpha}\left(\frac{v}{h_2}\right)-\frac{\partial}{\partial\beta}\left(\frac{u}{h_1}\right)\right] \end{aligned}\right\} \quad \ldots\ldots\ldots\ (12).$$

122. The stress-equations of equilibrium or small motion.

We have next to consider the expression of the stress-equations, referred to the same system of axes. Let δV be an element of volume contained by the three pairs of surfaces α and $\alpha+d\alpha$, β and $\beta+d\beta$, γ and $\gamma+d\gamma$, and Δ_1, Δ_2, Δ_3 the areas of the faces of this element, which lie in the tangent planes to α, β, γ respectively. Denote by P, Q, R, S, T, U the system of six stresses acting at any point α, β, γ, P being the stress on the face Δ_1 in the direction of dn_1, and S being the stress on the face Δ_2 in the direction of dn_3, or on the face Δ_3 in the direction of dn_2, and similarly for the others. Then the equation of motion of the element parallel to x_1 is

$$dx_1\frac{\partial}{\partial x_1}(P\Delta_1)+dy_1\frac{\partial}{\partial y_1}(U\Delta_2)+dz_1\frac{\partial}{\partial z_1}(T\Delta_3)+\rho X_1\delta V=\rho\frac{\partial^2u}{\partial t^2}\delta V\ldots(13),$$

X_1, Y_1, Z_1 being the components of the bodily force per unit of mass parallel to dn_1, dn_2, dn_3 respectively.

In the above equation $\delta V=d\alpha d\beta d\gamma/h_1h_2h_3$, and $\Delta_1=d\beta d\gamma/h_2h_3$, and so on.

Now $P\Delta_1$ is the x_1-component of a vector quantity whose other components are $U\Delta_1$ and $T\Delta_1$, viz., this quantity is the resultant stress across the surface-element Δ_1.

Hence

$$\begin{aligned} &\frac{\partial(P\Delta_1)}{\partial x_1}dx_1+\frac{\partial(P\Delta_1)}{\partial y_1}dy_1+\frac{\partial(P\Delta_1)}{\partial z_1}dz_1\\ &=\frac{\partial(P\Delta_1)}{\partial\alpha}d\alpha+\frac{\partial(P\Delta_1)}{\partial\beta}d\beta+\frac{\partial(P\Delta_1)}{\partial\gamma}d\gamma-U\Delta_1\delta\theta_3+T\Delta_1\delta\theta_2, \end{aligned}$$

so that remembering $dx_1 = d\alpha/h_1, \ldots$ we have

$$\frac{\partial(P\Delta_1)}{\partial x_1}dx_1 = d\alpha\left[\frac{\partial(P\Delta_1)}{\partial\alpha} + U\Delta_1 h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + T\Delta_1 h_3\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\right]$$

$$= d\alpha d\beta d\gamma\left[\frac{\partial}{\partial\alpha}\left(\frac{P}{h_2h_3}\right) + \frac{h_2U}{h_2h_3}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{h_3T}{h_2h_3}\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\right]$$

$$= d\alpha d\beta d\gamma\left[\frac{1}{h_2h_3}\frac{\partial P}{\partial\alpha} + \frac{P}{h_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right) + \frac{P}{h_3}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) + \frac{U}{h_3}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right.$$

$$\left. + \frac{T}{h_2}\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\right] \ldots\ldots\ldots\ldots\ldots(14).$$

Again $U\Delta_2$ is the x_1-component of a vector quantity whose other components are $Q\Delta_2$ and $S\Delta_2$, so that

$$\frac{\partial(U\Delta_2)}{\partial x_1}dx_1 + \frac{\partial(U\Delta_2)}{\partial y_1}dy_1 + \frac{\partial(U\Delta_2)}{\partial z_1}dz_1$$

$$= \frac{\partial(U\Delta_2)}{\partial\alpha}d\alpha + \frac{\partial(U\Delta_2)}{\partial\beta}d\beta + \frac{\partial(U\Delta_2)}{\partial\gamma}d\gamma - Q\Delta_2\delta\theta_3 + S\Delta_2\delta\theta_2.$$

Hence

$$\frac{\partial(U\Delta_2)}{\partial y_1}dy_1 = d\alpha d\beta d\gamma\left[\frac{1}{h_3h_1}\frac{\partial U}{\partial\beta} + \frac{U}{h_3}\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \frac{U}{h_1}\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right) - \frac{Q}{h_3}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right]$$

$\ldots\ldots\ldots\ldots(15).$

In like manner

$$\frac{\partial(T\Delta_3)}{\partial z_1}dz_1 = d\alpha d\beta d\gamma\left[\frac{1}{h_1h_2}\frac{\partial T}{\partial\gamma} + \frac{T}{h_1}\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right) + \frac{T}{h_2}\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right) - \frac{R}{h_2}\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)\right]$$

$\ldots\ldots\ldots\ldots(16).$

Hence multiplying throughout by $\dfrac{h_1h_2h_3}{d\alpha d\beta d\gamma}$ equation (13) becomes

$$\rho\left(\frac{\partial^2u}{\partial t^2} - X_1\right) = h_1\frac{\partial P}{\partial\alpha} + h_2\frac{\partial U}{\partial\beta} + h_3\frac{\partial T}{\partial\gamma} + h_2U\left[2h_1\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + h_3\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)\right]$$

$$+ h_3T\left[2h_1\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right) + h_2\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)\right]$$

$$+ h_1\left[h_2(P-Q)\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) + h_3(P-R)\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)\right] \ldots\ldots\ldots(17).$$

In like manner we may form the equations of motion parallel to y_1 and z_1; they are

$$\rho\left(\frac{\partial^2 v}{\partial t^2}-Y_1\right)=h_1\frac{\partial U}{\partial\alpha}+h_2\frac{\partial Q}{\partial\beta}+h_3\frac{\partial S}{\partial\gamma}+h_3S\left[2h_2\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)+h_1\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\right]$$
$$+h_1U\left[2h_2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)+h_3\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)\right]$$
$$+h_2\left[h_3(Q-R)\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)+h_1(Q-P)\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right]\ldots\ldots(18),$$

and

$$\rho\left(\frac{\partial^2 w}{\partial t^2}-Z_1\right)=h_1\frac{\partial T}{\partial\alpha}+h_2\frac{\partial S}{\partial\beta}+h_3\frac{\partial R}{\partial\gamma}+h_1T\left[2h_3\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)+h_2\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)\right]$$
$$+h_2S\left[2h_3\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)+h_1\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)\right]$$
$$+h_3\left[h_1(R-P)\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)+h_2(R-Q)\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)\right]\ldots\ldots(19).$$

These equations were first given by Lamé who obtained them by direct transformation from the Cartesian equations.

From these equations we may obtain the differential equations in terms of u, v, w by means of the stress-strain relations. When the solid is isotropic we have

$$\left.\begin{aligned} &P=\lambda\Delta+2\mu e, \quad Q=\lambda\Delta+2\mu f, \quad R=\lambda\Delta+2\mu g\\ &S=\mu a, \qquad\quad T=\mu b, \qquad\quad U=\mu c\\ &\qquad\qquad \Delta=e+f+g \end{aligned}\right\}\ldots\ldots(20).$$

We shall however be able to obtain the equations for u, v, w as well as equations (17), (18), (19) more directly by using the energy-method explained in the last chapter.

123. Application of the energy-method.

To obtain the stress-equations (17), (18), (19) by variation of the energy-function, we set out from the known result (art. 64) that all the equations and conditions are included in the general equation

$$\iiint\delta W\,dx\,dy\,dz$$
$$=\iiint\left[\left(\rho X_1-\rho\frac{\partial^2 u}{\partial t^2}\right)\delta u+\left(\rho Y_1-\rho\frac{\partial^2 v}{\partial t^2}\right)\delta v+\left(\rho Z_1-\rho\frac{\partial^2 w}{\partial t^2}\right)\delta w\right]dx\,dy\,dz$$
$$+\iint(F\delta u+G\delta v+H\delta w)\,dS\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

With orthogonal coordinates α, β, γ we must replace $dxdydz$ herein by the element of volume expressed by $d\alpha d\beta d\gamma/(h_1h_2h_3)$.

We have to evaluate

$$\iiint \delta W \frac{d\alpha d\beta d\gamma}{h_1h_2h_3},$$

where
$$\delta W = \frac{\partial W}{\partial e}\delta e + \frac{\partial W}{\partial f}\delta f + \ldots + \frac{\partial W}{\partial c}\delta c \quad \ldots\ldots\ldots(21^*).$$

Also we have, by (10),

$$\delta e = h_1 \frac{\partial \delta u}{\partial \alpha} + h_1h_2 \frac{\partial}{\partial \beta}\left(\frac{1}{h_1}\right)\delta v + h_1h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_1}\right)\delta w \ldots\ldots(22).$$

Thus we shall have to evaluate terms of the form

$$\iiint h_1 \frac{\partial \delta u}{\partial \alpha}\frac{\partial W}{\partial e}\frac{d\alpha d\beta d\gamma}{h_1h_2h_3} \quad \ldots\ldots\ldots\ldots\ldots(23).$$

Now if χ, ϕ be any functions of α, β, γ, we have

$$\chi \frac{\partial \delta\phi}{\partial \alpha} = \frac{\partial}{\partial \alpha}(\chi \delta\phi) - \frac{\partial \chi}{\partial \alpha}\delta\phi \quad \ldots\ldots\ldots\ldots(24).$$

Hence the above term (23) is

$$\iiint \left[\frac{\partial}{\partial \alpha}\left\{\frac{1}{h_2h_3}\frac{\partial W}{\partial e}\delta u\right\} - \delta u \frac{\partial}{\partial \alpha}\left\{\frac{1}{h_2h_3}\frac{\partial W}{\partial e}\right\}\right] d\alpha d\beta d\gamma \ldots(25).$$

Again if ξ be any uniform function of α, β, γ and the integration extend to all points within a closed surface S

$$\iiint \frac{\partial \xi}{\partial \alpha} d\alpha d\beta d\gamma = \iint h_2h_3 \xi l dS \quad \ldots\ldots\ldots\ldots(26),$$

where l, m, n are the cosines of the angles which the normal to S, drawn outwards, makes with the normals to α, β, γ, at any point of S.

Thus the term in $\frac{\partial \delta u}{\partial \alpha}$ is transformed into the sum of a volume-integral and a surface-integral. In like manner all the terms containing differential coefficients of u, v, w may be transformed each into the sum of a volume-integral and a surface-integral.

In the expression of

$$\iiint \delta W \frac{d\alpha d\beta d\gamma}{h_1h_2h_3}$$

we shall collect the terms containing δu.

The volume-integral is

$$\iiint\Big[-\frac{\partial}{\partial\alpha}\Big\{\frac{1}{h_2h_3}\frac{\partial W}{\partial e}\Big\}+\frac{1}{h_3}\frac{\partial}{\partial\alpha}\Big(\frac{1}{h_2}\Big)\frac{\partial W}{\partial f}+\frac{1}{h_2}\frac{\partial}{\partial\alpha}\Big(\frac{1}{h_3}\Big)\frac{\partial W}{\partial g}$$
$$-\frac{\partial}{\partial\gamma}\Big\{\frac{1}{h_1h_2}\frac{\partial W}{\partial b}\Big\}-\frac{1}{h_2}\frac{\partial}{\partial\gamma}\Big(\frac{1}{h_1}\Big)\frac{\partial W}{\partial b}$$
$$-\frac{\partial}{\partial\beta}\Big\{\frac{1}{h_3h_1}\frac{\partial W}{\partial c}\Big\}-\frac{1}{h_3}\frac{\partial}{\partial\beta}\Big(\frac{1}{h_1}\Big)\frac{\partial W}{\partial c}\Big]\,\delta u\,d\alpha d\beta d\gamma.$$

The surface-integral is

$$\iint\Big(l\frac{\partial W}{\partial e}+m\frac{\partial W}{\partial c}+n\frac{\partial W}{\partial b}\Big)\,\delta u dS.$$

The sum of all the terms thus obtained has to be equated to

$$\iiint\Big[\Big(\rho X_1-\rho\frac{\partial^2 u}{\partial t^2}\Big)\,\delta u+\ldots+\ldots\Big]\frac{d\alpha d\beta d\gamma}{h_1h_2h_3}+\iint(F\delta u+G\delta v+H\delta w)\,dS.$$

By equating the terms in δu in the volume-integrals we obtain the equation

$$h_1h_2h_3\Big[\frac{\partial}{\partial\alpha}\Big(\frac{1}{h_2h_3}\frac{\partial W}{\partial e}\Big)+\frac{\partial}{\partial\beta}\Big(\frac{1}{h_3h_1}\frac{\partial W}{\partial c}\Big)+\frac{\partial}{\partial\gamma}\Big(\frac{1}{h_1h_2}\frac{\partial W}{\partial b}\Big)\Big]$$
$$-h_1h_2\Big[\frac{\partial}{\partial\alpha}\Big(\frac{1}{h_2}\Big)\frac{\partial W}{\partial f}-\frac{\partial}{\partial\beta}\Big(\frac{1}{h_1}\Big)\frac{\partial W}{\partial c}\Big]$$
$$-h_1h_3\Big[\frac{\partial}{\partial\alpha}\Big(\frac{1}{h_3}\Big)\frac{\partial W}{\partial g}-\frac{\partial}{\partial\gamma}\Big(\frac{1}{h_1}\Big)\frac{\partial W}{\partial b}\Big]+\rho X_1=\rho\frac{\partial^2 u}{\partial t^2}\quad\ldots\ldots(27).$$

This is identical with equation (17), and the equations corresponding to (18) and (19) can be written down by symmetry.

In like manner by equating the terms in δu, δv, δw in the surface-integrals we obtain the boundary-conditions

$$\left.\begin{aligned}lP+mU+nT&=F,\\ lU+mQ+nS&=G,\\ lT+mS+nR&=H\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(28).$$

The strain-equations can be found as before by substituting for the stresses their expression in terms of the strains, or by beginning with the expression for W in terms of strain-components.

124. Strain-equations for isotropic solid.

In the case of an isotropic solid, the strain-equations can be put into a particularly simple form.

The energy per unit volume W is given by the equation

$$2W = (\lambda + 2\mu)\Delta^2 + \mu[a^2 + b^2 + c^2 - 4fg - 4ge - 4ef] \ldots (29),$$

where Δ and $a^2 + b^2 + c^2 - 4fg - 4ge - 4ef$ are invariants.

Now we have proved (art. 11) that $\varpi_1^2 + \varpi_2^2 + \varpi_3^2$ is also an invariant. We have the identities

$$\left.\begin{aligned} a^2 - 4fg &= 4\varpi_1^2 + 4\left(\frac{\partial w}{\partial y_1}\frac{\partial v}{\partial z_1} - \frac{\partial w}{\partial z_1}\frac{\partial v}{\partial y_1}\right), \\ b^2 - 4ge &= 4\varpi_2^2 + 4\left(\frac{\partial u}{\partial z_1}\frac{\partial w}{\partial x_1} - \frac{\partial u}{\partial x_1}\frac{\partial w}{\partial z_1}\right), \\ c^2 - 4ef &= 4\varpi_3^2 + 4\left(\frac{\partial v}{\partial x_1}\frac{\partial u}{\partial y_1} - \frac{\partial v}{\partial y_1}\frac{\partial u}{\partial x_1}\right) \end{aligned}\right\} \ldots\ldots\ldots (30).$$

Hence also the quantity

$$\left(\frac{\partial w}{\partial y_1}\frac{\partial v}{\partial z_1} - \frac{\partial w}{\partial z_1}\frac{\partial v}{\partial y_1}\right) + \left(\frac{\partial u}{\partial z_1}\frac{\partial w}{\partial x_1} - \frac{\partial u}{\partial x_1}\frac{\partial w}{\partial z_1}\right) + \left(\frac{\partial v}{\partial x_1}\frac{\partial u}{\partial y_1} - \frac{\partial v}{\partial y_1}\frac{\partial u}{\partial x_1}\right)$$

is an invariant for orthogonal transformations.

Now take a fixed system of axes of x, y, z, and let U_1, V_1, W_1 be component displacements referred to this system of axes, then according to the theorem just quoted

$$\left(\frac{\partial w}{\partial y_1}\frac{\partial v}{\partial z_1} - \frac{\partial w}{\partial z_1}\frac{\partial v}{\partial y_1}\right) + \ldots + \ldots = \left(\frac{\partial W_1}{\partial y}\frac{\partial V_1}{\partial z} - \frac{\partial W_1}{\partial z}\frac{\partial V_1}{\partial y}\right) + \ldots + \ldots$$

$\ldots\ldots\ldots (31),$

where the fixed system (x, y, z) is quite independent of the directions of the (x_1, y_1, z_1) axes, which are the normals to the surfaces α, β, γ at any point.

Thus in varying the energy we have to find the variation of the functions Δ, ϖ_1, ϖ_2, ϖ_3, and of such quantities as

$$\iiint\left(\frac{\partial W_1}{\partial y}\frac{\partial V_1}{\partial z} - \frac{\partial W_1}{\partial z}\frac{\partial V_1}{\partial y}\right) dxdydz.$$

Now the variation of such a quantity as this last can contribute surface-terms only; for

$$\iiint\left(\frac{\partial \delta W_1}{\partial y}\frac{\partial V_1}{\partial z} + \frac{\partial W_1}{\partial y}\frac{\partial \delta V_1}{\partial z} - \frac{\partial \delta W_1}{\partial z}\frac{\partial V_1}{\partial y} - \frac{\partial W_1}{\partial z}\frac{\partial \delta V_1}{\partial y}\right) dxdydz$$

$$= \iiint\left[\frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z}\delta W_1\right) - \delta W_1\frac{\partial^2 V_1}{\partial y \partial z}\right.$$

$$\left. - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial y}\delta W_1\right) + \delta W_1\frac{\partial^2 V_1}{\partial z \partial y}\right] dxdydz$$

$$+ \text{terms containing } \delta V_1.$$

The parts $\frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z}\delta W_1\right)$ and $-\frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial y}\delta W_1\right)$ contribute only surface-integrals, and the other parts vanish identically. The terms in δV_1 may similarly be shewn to contribute surface-integrals only.

Hence the volume-integral part of

$$\iiint \delta W \, dx dy dz$$

is the same as that of

$$\iiint \delta \left\{\tfrac{1}{2}(\lambda + 2\mu)\Delta^2 + 2\mu(\varpi_1^2 + \varpi_2^2 + \varpi_3^2)\right\} \frac{d\alpha d\beta d\gamma}{h_1 h_2 h_3} \quad \ldots(32),$$

and we can obtain the equations of equilibrium or small motion in terms of u, v, w by variation of this integral.

The term in δu proceeding from the variation of the term in Δ^2 is

$$\iiint (\lambda + 2\mu)\Delta h_1 h_2 h_3 \frac{\partial}{\partial \alpha}\left(\frac{\delta u}{h_2 h_3}\right) \frac{d\alpha d\beta d\gamma}{h_1 h_2 h_3},$$

and this is

$$-\iiint (\lambda + 2\mu)\delta u \frac{\partial \Delta}{\partial \alpha} \frac{d\alpha d\beta d\gamma}{h_2 h_3} + \text{a surface-integral} \ldots(33).$$

There is no term in δu proceeding from the variation of ϖ_1.

The term in δu proceeding from the variation of ϖ_2 is

$$\iiint 2\mu \varpi_2 h_3 h_1 \frac{\partial}{\partial \gamma}\left(\frac{\delta u}{h_1}\right) \frac{d\alpha d\beta d\gamma}{h_1 h_2 h_3},$$

and this is

$$-\iiint 2\mu \frac{\delta u}{h_1} \frac{\partial}{\partial \gamma}\left(\frac{\varpi_2}{h_2}\right) d\alpha d\beta d\gamma + \text{a surface-integral} \ldots\ldots(34).$$

The term in δu proceeding from the variation of ϖ_3 is in like manner

$$\iiint 2\mu \frac{\delta u}{h_1} \frac{\partial}{\partial \beta}\left(\frac{\varpi_3}{h_3}\right) d\alpha d\beta d\gamma + \text{a surface-integral} \ldots\ldots(35).$$

Hence the volume-integral part of $\iiint \delta W \, dx dy dz$ is

$$-\iiint \delta u \left\{(\lambda + 2\mu) h_1 \frac{\partial \Delta}{\partial \alpha} - 2\mu h_2 h_3 \frac{\partial}{\partial \beta}\left(\frac{\varpi_3}{h_3}\right) + 2\mu h_2 h_3 \frac{\partial}{\partial \gamma}\left(\frac{\varpi_2}{h_2}\right)\right\} \frac{d\alpha d\beta d\gamma}{h_1 h_2 h_3}$$

$-$ similar terms in δv and δw $\ldots\ldots(36).$

Thus we obtain the three equations of equilibrium

$$\left.\begin{aligned}
(\lambda+2\mu)\,h_1\frac{\partial\Delta}{\partial\alpha}-2\mu h_2h_3\frac{\partial}{\partial\beta}\left(\frac{\varpi_3}{h_3}\right)+2\mu h_2h_3\frac{\partial}{\partial\gamma}\left(\frac{\varpi_2}{h_2}\right)+\rho X_1=\rho\frac{\partial^2u}{\partial t^2},\\
(\lambda+2\mu)\,h_2\frac{\partial\Delta}{\partial\beta}-2\mu h_3h_1\frac{\partial}{\partial\gamma}\left(\frac{\varpi_1}{h_1}\right)+2\mu h_3h_1\frac{\partial}{\partial\alpha}\left(\frac{\varpi_3}{h_3}\right)+\rho Y_1=\rho\frac{\partial^2v}{\partial t^2},\\
(\lambda+2\mu)\,h_3\frac{\partial\Delta}{\partial\gamma}-2\mu h_1h_2\frac{\partial}{\partial\alpha}\left(\frac{\varpi_2}{h_2}\right)+2\mu h_1h_2\frac{\partial}{\partial\beta}\left(\frac{\varpi_1}{h_1}\right)+\rho Z_1=\rho\frac{\partial^2w}{\partial t^2}
\end{aligned}\right\}\quad(37).$$

Equations equivalent to these were first given by Lamé, who obtained them by direct transformation from the Cartesian equations.

The boundary-conditions are, as in ch. III. (15),

$$\left.\begin{aligned}
F=\ l\lambda\Delta+2\mu\left(\frac{\partial u}{\partial n'}+m\varpi_3-n\varpi_2\right),\\
G=m\lambda\Delta+2\mu\left(\frac{\partial v}{\partial n'}+n\varpi_1-l\varpi_3\right),\\
H=\ n\lambda\Delta+2\mu\left(\frac{\partial w}{\partial n'}+l\varpi_2-m\varpi_1\right)
\end{aligned}\right\}\quad\ldots\ldots\ldots(38),$$

where F, G, H are the surface-tractions estimated in the directions of the normals to the surfaces (α, β, γ) which meet in any point of the bounding surface, (l, m, n) are the cosines of the angles which the normal to the bounding surface drawn outwards makes with these three normals, and dn' is the element of the normal to the bounding surface.

125. Systems of Orthogonal Coordinates.

Among particular systems of curvilinear orthogonal coordinates, we may mention

1°. *Polar coordinates.*

The coordinates are the colatitude θ, the longitude ϕ, and the radius vector r, and the surfaces are

coaxal cones $\theta=$ const.,

planes through the fixed axis $\phi=$ const.,

concentric spheres $r=$ const.

If we take $\theta=\alpha$, $\phi=\beta$, $r=\gamma$ the values of the h's are

$$h_1=1/r,\quad h_2=\operatorname{cosec}\theta/r,\quad h_3=1.$$

2°. *Elliptic coordinates.*

The surfaces are confocal quadrics. For an account of the system the reader is referred to Salmon's *Geometry of Three Dimensions*, ch. XII. sect. IV., and for applications in the theory of Potential and in Hydrodynamics to Heine's *Handbuch der Kugelfunctionen*, and to M. Poincaré's memoir in the *Acta Mathematica*, vol. VII. There are at present no applications of importance in the theory of Elasticity.

3°. *Cylindrical systems* derived by means of *conjugate functions.*

Suppose $$\alpha + \iota\beta = f(x + \iota y),$$

so that α and β are the real and imaginary parts of a function of a complex variable in the plane x, y. Then it is well known that the curves $\alpha = \text{const.}$, $\beta = \text{const.}$ cut at right angles. It follows that we may take $\alpha = \text{const.}$ and $\beta = \text{const.}$ for two families of cylindrical surfaces cutting at right angles, and the planes $z = \text{const.}$ will cut each of them at right angles. Hence α, β, z form a system of orthogonal surfaces. Such systems ought to prove useful in the solution of problems relating to bodies with cylindrical boundaries.

4°. *Systems of revolution.*

Let $\varpi^2 = x^2 + y^2$, and suppose $\alpha + \iota\beta = f(z + \iota\varpi)$, then in the plane z, ϖ the curves α, β cut at right angles. If this plane be made to turn about the axis z, the surfaces $\alpha = \text{const.}$, $\beta = \text{const.}$, and the planes $\phi = \text{const.}$ drawn through the axis z are a system of orthogonal surfaces. We shall consider some examples of the application of such systems in our subject later. For other applications the reader is referred to Mr Basset's *Hydrodynamics*, vol. II., to a paper by Mr Bryan in *Phil. Trans. R. S.* 1888, and to Mr Hicks's memoirs on *Toroidal Functions* and on *Vortex Motion* in *Phil. Trans. R. S.* 1881, 1884, 1885.

We leave to the reader the verification of the following results for polar coordinates, the displacements u, v, w being in the directions of the meridian, parallel, and radius through the point (θ, ϕ, r):

1°. The strains are

$$e = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r},$$

$$f = \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi} + \frac{u}{r}\cot\theta + \frac{w}{r},$$

$$g = \frac{\partial w}{\partial r},$$

$$a = \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r},$$

$$b = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{u}{r},$$

$$c = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{v}{r}\cot\theta.$$

2°. The cubical dilatation is given by the equation

$$\Delta = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(ur\sin\theta) + \frac{\partial}{\partial\phi}(vr) + \frac{\partial}{\partial r}(wr^2\sin\theta)\right].$$

3°. The three rotations are given by the equations

$$2\varpi_1 = \frac{1}{r\sin\theta}\left[\frac{\partial w}{\partial\phi} - \frac{\partial}{\partial r}(vr\sin\theta)\right],$$

$$2\varpi_2 = \frac{1}{r}\left[\frac{\partial}{\partial r}(ur) - \frac{\partial w}{\partial\theta}\right],$$

$$2\varpi_3 = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(vr\sin\theta) - \frac{\partial}{\partial\phi}(ur)\right].$$

4°. The stress-equations are

$$\frac{1}{r}\frac{\partial P}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi} + \frac{\partial T}{\partial r} + \frac{3T + (P-Q)\cot\theta}{r} = \rho\left(\frac{\partial^2 u}{\partial t^2} - X_1\right),$$

$$\frac{1}{r}\frac{\partial U}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial Q}{\partial\phi} + \frac{\partial S}{\partial r} + \frac{3S + 2U\cot\theta}{r} = \rho\left(\frac{\partial^2 v}{\partial t^2} - Y_1\right),$$

$$\frac{1}{r}\frac{\partial T}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial S}{\partial\phi} + \frac{\partial R}{\partial r} + \frac{T\cot\theta + 2R - P - Q}{r} = \rho\left(\frac{\partial^2 w}{\partial t^2} - Z_1\right),$$

in which, with Prof. Pearson's notation (see art. 49) $P = \widehat{\theta\theta}$, $Q = \widehat{\phi\phi}$, $R = \widehat{rr}$, $S = \widehat{\phi r}$, $T = \widehat{r\theta}$, $U = \widehat{\theta\phi}$, and X_1, Y_1, Z_1 are the components of the bodily force per unit mass in the directions θ, ϕ, r.

5°. The equations of motion in terms of displacements[1] for an isotropic body are

$$(\lambda+2\mu)\sin\theta\frac{\partial\Delta}{\partial\theta}-2\mu\frac{\partial\varpi_3}{\partial\phi}+2\mu\sin\theta\frac{\partial}{\partial r}(r\varpi_2)=\rho r\sin\theta\left(\frac{\partial^2 u}{\partial t^2}-X_1\right),$$

$$(\lambda+2\mu)\operatorname{cosec}\theta\frac{\partial\Delta}{\partial\phi}-2\mu\frac{\partial}{\partial r}(r\varpi_1)+2\mu\frac{\partial\varpi_3}{\partial\theta}=\rho r\left(\frac{\partial^2 v}{\partial t^2}-Y_1\right),$$

$$(\lambda+2\mu)r^2\sin\theta\frac{\partial\Delta}{\partial r}-2\mu r\frac{\partial}{\partial\theta}(\varpi_2\sin\theta)+2\mu r\frac{\partial\varpi_1}{\partial\phi}=\rho r^2\sin\theta\left(\frac{\partial^2 w}{\partial t^2}-Z_1\right).$$

In like manner in cylindrical coordinates, r, θ, z, the displacements being u along the radius, v along the tangent to the circular section, and w along the generator, we have the following results:

1°. The strains are

$$e=\frac{\partial u}{\partial r}, \qquad a=\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial z},$$

$$f=\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}, \qquad b=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r},$$

$$g=\frac{\partial w}{\partial z}, \qquad c=\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}-\frac{v}{r}.$$

2°. The cubical dilatation is given by the equation

$$\Delta=\frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z}.$$

3°. The three rotations are given by the equations

$$2\varpi_1=\frac{1}{r}\left[\frac{\partial w}{\partial\theta}-\frac{\partial}{\partial z}(rv)\right],$$

$$2\varpi_2=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r},$$

$$2\varpi_3=\frac{1}{r}\left[\frac{\partial}{\partial r}(rv)-\frac{\partial u}{\partial\theta}\right].$$

[1] For applications of polar coordinates the reader is referred to Mr Chree's paper on 'The equations of an isotropic solid in polar and cylindrical coordinates', *Camb. Phil. Soc. Trans.* XIV., 1889.

4°. The stress-equations are

$$\frac{\partial P}{\partial r}+\frac{1}{r}\frac{\partial U}{\partial \theta}+\frac{\partial T}{\partial z}+\frac{P-Q}{r}=\rho\left(\frac{\partial^2 u}{\partial t^2}-X_1\right),$$

$$\frac{\partial U}{\partial r}+\frac{1}{r}\frac{\partial Q}{\partial \theta}+\frac{\partial S}{\partial z}+\frac{2U}{r}=\rho\left(\frac{\partial^2 v}{\partial t^2}-Y_1\right),$$

$$\frac{\partial T}{\partial r}+\frac{1}{r}\frac{\partial S}{\partial \theta}+\frac{\partial R}{\partial z}+\frac{T}{r}=\rho\left(\frac{\partial^2 w}{\partial t^2}-Z_1\right),$$

in which, with Prof. Pearson's notation, $P=\widehat{rr}$, $Q=\widehat{\theta\theta}$, $R=\widehat{zz}$, $S=\widehat{\theta z}$, $T=\widehat{zr}$, $U=\widehat{r\theta}$, and X_1, Y_1, Z_1 are the components of the bodily force per unit mass in the directions r, θ, z.

5°. The equations in terms of the displacements for an isotropic body are

$$(\lambda+2\mu)\,r\frac{\partial \Delta}{\partial r}-2\mu\frac{\partial \varpi_3}{\partial \theta}+2\mu r\frac{\partial \varpi_2}{\partial z}=\rho r\left(\frac{\partial^2 u}{\partial t^2}-X_1\right),$$

$$(\lambda+2\mu)\frac{1}{r}\frac{\partial \Delta}{\partial \theta}-2\mu\frac{\partial \varpi_1}{\partial z}+2\mu\frac{\partial \varpi_3}{\partial r}=\rho\left(\frac{\partial^2 v}{\partial t^2}-Y_1\right),$$

$$(\lambda+2\mu)\,r\frac{\partial \Delta}{\partial z}-2\mu\frac{\partial}{\partial r}(r\varpi_2)+2\mu\frac{\partial \varpi_1}{\partial \theta}=\rho r\left(\frac{\partial^2 w}{\partial t^2}-Z_1\right).$$

126. Radial Strain. Polar Coordinates.

We proceed to consider the very simple example of purely radial strain of isotropic matter referred to polar coordinates. For this it is simpler to proceed by a different method.

Suppose the displacement of a point to be U' along the radius, and zero in any other direction, then the displacements parallel to x, y, z are

$$u=U'\frac{x}{r},\quad v=U'\frac{y}{r},\quad w=U'\frac{z}{r}\qquad\text{.....(39),}$$

where U' is a function of r.

The strains e, f, g, referred to the fixed axes of x, y, z, are

$$\left.\begin{aligned} e&=\frac{\partial u}{\partial x}=\frac{\partial U'}{\partial r}\frac{x^2}{r^2}+\frac{U'}{r}-\frac{U'x^2}{r^3},\\ f&=\frac{\partial v}{\partial y}=\frac{\partial U'}{\partial r}\frac{y^2}{r^2}+\frac{U'}{r}-\frac{U'y^2}{r^3},\\ g&=\frac{\partial w}{\partial z}=\frac{\partial U'}{\partial r}\frac{z^2}{r^2}+\frac{U'}{r}-\frac{U'z^2}{r^3}\end{aligned}\right\}\qquad\text{.....(40),}$$

so that the cubical dilatation is given by the equation

$$\Delta = \frac{\partial U'}{\partial r} + \frac{2U'}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41).$$

The rotations are ϖ_1, ϖ_2, ϖ_3, where

$$2\varpi_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial U'}{\partial r}\left(\frac{yz}{r^2} - \frac{yz}{r^2}\right) + U'\left(-\frac{z}{r}\frac{y}{r^2} + \frac{y}{r}\frac{z}{r^2}\right) = 0,$$

so $2\varpi_2 = 0$ and $2\varpi_3 = 0$, as indeed is physically obvious.

In the equations of small motion the bodily forces must reduce to a purely radial force, R say, and

$$X = R\frac{x}{r}, \quad Y = R\frac{y}{r}, \quad Z = R\frac{z}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (42).$$

The equations of small motion are, by (13) of ch. III.,

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial x} + \rho R\frac{x}{r} = \rho\frac{\partial^2}{\partial t^2}\left(U'\frac{x}{r}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (43),$$

and two similar equations.

Multiplying these by $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, and adding, we have

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial r} + \rho R = \rho\frac{\partial^2 U'}{\partial t^2},$$

or

$$(\lambda + 2\mu)\frac{\partial}{\partial r}\left(\frac{\partial U'}{\partial r} + \frac{2U'}{r}\right) + \rho R = \rho\frac{\partial^2 U'}{\partial t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (44).$$

To estimate the traction across a concentric sphere, suppose this traction to be a tension T along the radius outwards; the component tractions are Tx/r, Ty/r, Tz/r. Hence equations (15) of ch. III. become three such as

$$T\frac{x}{r} = \frac{x}{r}\lambda\Delta + 2\mu\frac{\partial}{\partial r}\left(U'\frac{x}{r}\right),$$

or

$$T\frac{x}{r} = \frac{x}{r}\left[(\lambda + 2\mu)\frac{\partial U'}{\partial r} + 2\lambda\frac{U'}{r}\right].$$

Thus

$$T = (\lambda + 2\mu)\frac{\partial U'}{\partial r} + 2\lambda\frac{U'}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (45).$$

This is the radial traction per unit area across any element of a concentric sphere of radius r.

We shall now consider some examples of these formulae, and, as we do not require U to denote a component of stress, we shall suppress the accent on U'.

127. Compression of a sphere due to its own gravitation[1].

Let a be the radius of the bounding surface in the strained state, and let the bodily force at a distance r be $-gr/a$.

The equation of equilibrium is

$$(\lambda+2\mu)\frac{d}{dr}\left(\frac{dU}{dr}+\frac{2U}{r}\right)-g\rho\frac{r}{a}=0.$$

Putting $g\rho=10H(\lambda+2\mu)a$, the equation becomes

$$r^2\frac{d^2U}{dr^2}+2r\frac{dU}{dr}-2U=10Hr^3.$$

The complete primitive of this equation is

$$U=Ar+\frac{B}{r^2}+Hr^3,$$

where A and B are arbitrary constants.

As U must be finite at the centre of the sphere, we must put $B=0$, and thus

$$U=Ar+Hr^3.$$

Suppose the surface free, then $T=0$ when $r=a$, or

$$(\lambda+2\mu)(A+3Ha^2)+2\lambda(A+Ha^2)=0,$$

hence
$$A=-Ha^2\frac{5\lambda+6\mu}{3\lambda+2\mu},$$

so that
$$U=-\tfrac{1}{10}\frac{g\rho ar}{\lambda+2\mu}\left(\frac{5\lambda+6\mu}{3\lambda+2\mu}-\frac{r^2}{a^2}\right) \quad \text{.....(46).}$$

Writing this in terms of Poisson's ratio σ, where

$$\sigma=\tfrac{1}{2}\lambda/(\lambda+\mu),$$

we have

$$U=-\tfrac{1}{10}\frac{g\rho ar}{\lambda+2\mu}\left(\frac{3-\sigma}{1+\sigma}-\frac{r^2}{a^2}\right) \quad \text{.....(47).}$$

The displacement is everywhere towards the centre, since by art. (28) $3-\sigma>1+\sigma$. The radial contraction $-dU/dr$ is

$$\tfrac{1}{10}\frac{g\rho a}{\lambda+2\mu}\left(\frac{3-\sigma}{1+\sigma}-\frac{3r^2}{a^2}\right) \quad \text{.....(48),}$$

so that the parts of the radii that lie within the sphere

$$r=a\sqrt{\{(3-\sigma)/(3+3\sigma)\}} \quad \text{.....(49)}$$

are contracted, and the parts that lie outside this sphere are extended.

[1] For further details in regard to this problem and those in arts. 128—130 the reader is referred to Mr Chree's paper quoted on p. 216.

The greatest extension has place at the surface, and is equal to

$$\tfrac{2}{5}\frac{g\rho a}{\lambda+2\mu}\frac{\sigma}{1+\sigma} \qquad \text{.....(50).}$$

According to the theory of Poncelet and Saint-Venant (art. 57) the sphere will be certainly unable to resist the strain arising from its own gravitation if the breaking stress T_0 of the material be less than $\frac{2}{5}E\frac{g\rho a}{\lambda+2\mu}\frac{\sigma}{1+\sigma}$, or the condition of safety is

$$T_0 > \tfrac{2}{5}\mu\frac{3\lambda+2\mu}{\lambda+\mu}\frac{\sigma}{1+\sigma}\frac{g\rho a}{\lambda+2\mu} \qquad \text{.....(51).}$$

Supposing, with Poisson, $\lambda=\mu$ and $\sigma=\frac{1}{4}$, this is $T_0 > \frac{1}{15}g\rho a$. For a sphere of the same size and mass as the Earth, this is greater than 237×10^6 grammes' weight per square centimetre, and the solution is not applicable to such a body.

There is another difficulty in the application of the result to the case of the Earth. The necessary limitation to the mathematical theory is that the strain found from it must always be "small". Now we found at the surface an extension

$$\tfrac{2}{5}\frac{g\rho a}{\lambda+2\mu}\frac{\sigma}{1+\sigma},$$

and this cannot be treated as a small quantity unless $g\rho a\sigma/(\lambda+2\mu)$ can be so treated. For a sphere of the size and density of the earth $g\rho a$ is about 3585×10^6 grammes' weight per square centimetre, which is greater than any modulus of any known homogeneous isotropic material, and for any such material it is clear that $g\rho a/(\lambda+2\mu)$ cannot be a small fraction.

In case the material be approximately incompressible so that λ is very great compared with μ we can have $g\rho a/(\lambda+2\mu)$ a small fraction of the order of strains usually considered. In any other case[1] what the work shews is that a sphere of the size and mass of the earth, homogeneous, and possessed of finite and comparable moduluses of rigidity and compression equal to those of any known material, could not exist. If such a solid existed for an instant, finite motions would ensue accompanied by large permanent sets.

[1] If σ be small the extension (50) is small, but (48) shews that large strains would exist in the interior.

128. Spherical Shell under internal and external Pressures.

As an example of equilibrium under surface-tractions, consider the case of a spherical shell, whose outer and inner surfaces are subjected to hydrostatic pressure.

Let r_0, r_1 be the radii of the outer and inner surfaces,

p_0, p_1 the pressures on them.

Then, when $r = r_0,\ T = -p_0,$

and, when $r = r_1,\ T = -p_1.$

The general solution of the differential equation of equilibrium

$$\frac{d}{dr}\left(\frac{dU}{dr} + \frac{2U}{r}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (52)$$

is $$U = Ar + \frac{B}{r^2}.$$

The radial stress at any point is

$$(\lambda + 2\mu)\left(A - \frac{2B}{r^3}\right) + 2\lambda\left(A + \frac{B}{r^3}\right),$$

thus $$(3\lambda + 2\mu)A - \frac{4\mu B}{r_0^3} = -p_0,$$

$$(3\lambda + 2\mu)A - \frac{4\mu B}{r_1^3} = -p_1;$$

hence $$B\left(\frac{1}{r_1^3} - \frac{1}{r_0^3}\right) = \frac{p_1 - p_0}{4\mu},$$

and

$$(3\lambda + 2\mu)A\left(\frac{1}{r_1^3} - \frac{1}{r_0^3}\right) = -\frac{p_0}{r_1^3} + \frac{p_1}{r_0^3};$$

from which

$$U = \frac{1}{3\lambda + 2\mu}\frac{p_1 r_1^3 - p_0 r_0^3}{r_0^3 - r_1^3} r + \frac{1}{4\mu}\frac{r_0^3 r_1^3 (p_1 - p_0)}{r_0^3 - r_1^3}\frac{1}{r^2} \quad \ldots\ldots (53).$$

In particular if $p_0 = 0,\ p_1 = p$ we have a spherical envelope strained by internal pressure. The displacement is

$$\frac{p r_1^3 r}{r_0^3 - r_1^3}\left[\frac{1}{3\lambda + 2\mu} + \frac{1}{4\mu}\frac{r_0^3}{r^3}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (54).$$

The radial extension at any point is

$$\frac{p r_1^3}{r_0^3 - r_1^3}\left[\frac{1}{3\lambda + 2\mu} - \frac{1}{2\mu}\frac{r_0^2}{r^2}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (55),$$

which is greatest at the inner surface.

The extension of any line perpendicular to the radius is

$$\frac{pr_1^3}{r_0^3 - r_1^3}\left(\frac{1}{3\lambda + 2\mu} + \frac{1}{4\mu}\frac{r_0^3}{r^3}\right) \text{.....(56)},$$

which is also greatest at the inner surface, and its value there is the greatest principal extension.

According to the theory of Poncelet and Saint-Venant (art. 57) the spherical envelope will be certainly unsafe if its breaking stress be less than the product of E and the above expression (56).

If we take $\lambda = \mu$ this condition becomes

$$T_0 < p\,\frac{4r_1^3 + 5r_0^3}{8\,(r_0^3 - r_1^3)} \text{.....(57)},$$

where T_0 is the breaking stress of the material.

If the envelope be of small thickness $2h$[1], and radius r, the condition of safety is

$$\Phi p < \tfrac{16}{3}\cdot\frac{h}{r}\cdot T_0 \text{.....(58)},$$

where Φ is the factor of safety. This gives the greatest safe pressure allowed by the theory referred to.

129. Vibrations of a spherical shell.

Suppose a shell whose internal radius is a, and external radius is b is vibrating freely, and that the displacement is purely radial.

The bodily forces and surface-tractions vanish; and the displacement U is determined by solving the differential equation

$$(\lambda + 2\mu)\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} - \frac{2}{r^2}U\right) = \rho\frac{\partial^2 U}{\partial t^2} \text{.....(59)},$$

subject to the conditions that

$$(\lambda + 2\mu)\frac{\partial U}{\partial r} + 2\lambda\frac{U}{r} = 0 \text{ ..... (60)},$$

when $r = a$ and when $r = b$.

Assume that U as a function of t is proportional to e^{ipt}, then the period of the small oscillations is $2\pi/p$, and, to determine U as a function of r, we have to solve the equation

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - \frac{2}{r^2}U + \frac{\rho p^2}{\lambda + 2\mu}U = 0.$$

[1] The factor 2 is inserted here as we shall find it always convenient in the theory of Thin Shells to represent the *half-thickness* of the shell by h. The spherical envelope is a "thin shell".

Writing

$$\left.\begin{aligned} rU &= ye^{ipt}, \\ \kappa^2(\lambda+2\mu) &= \rho p^2, \end{aligned}\right\}$$

the above equation becomes

$$\frac{d^2y}{dr^2}+\kappa^2 y=\frac{1.2}{r^2}\,y,$$

which is an integrable case of Riccati's equation. The complete primitive is

$$y=r^2\left(\frac{1}{r}\frac{d}{dr}\right)\left(\frac{A\sin\kappa r+B\cos\kappa r}{r}\right).$$

Hence with new arbitrary constants we may write

$$U=\frac{d}{d(\kappa r)}\left(\frac{A\sin\kappa r+B\cos\kappa r}{\kappa r}\right)e^{ipt},$$

or
$$U=\left[A\frac{\kappa r\cos\kappa r-\sin\kappa r}{(\kappa r)^2}-B\frac{\kappa r\sin\kappa r+\cos\kappa r}{(\kappa r)^2}\right]e^{ipt}\ldots(61).$$

From this

$$\frac{\partial U}{\partial r}=\frac{e^{ipt}}{\kappa^2 r^3}[A\{(2-\kappa^2r^2)\sin\kappa r-2\kappa r\cos\kappa r\} + B\{(2-\kappa^2r^2)\cos\kappa r+2\kappa r\sin\kappa r\}].$$

Hence at either surface we shall have

$$[(\lambda+2\mu)\{(2-\kappa^2r^2)\sin\kappa r-2\kappa r\cos\kappa r\}+2\lambda(\kappa r\cos\kappa r-\sin\kappa r)]A + [(\lambda+2\mu)\{(2-\kappa^2r^2)\cos\kappa r+2\kappa r\sin\kappa r\} - 2\lambda(\kappa r\sin\kappa r+\cos\kappa r)]B=0\ldots\ldots\ldots\ldots\ldots\ldots(62).$$

Writing

$$2\lambda/(\lambda+2\mu)=2-\nu\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

substituting successively a and b for r, and eliminating the ratio $A:B$, we obtain the equation

$$\frac{-\nu\kappa a+(\nu-\kappa^2a^2)\tan\kappa a}{\nu-\kappa^2a^2+\nu\kappa a\tan\kappa a}=\frac{-\nu\kappa b+(\nu-\kappa^2b^2)\tan\kappa b}{\nu-\kappa^2b^2+\nu\kappa b\tan\kappa b}\ldots(64).$$

This is an equation to find κ. When κ is found from this equation the type of vibration is given by (62), and the period $2\pi/p$ is

$$\frac{2\pi}{\kappa}\sqrt{\left(\frac{\rho}{\lambda+2\mu}\right)}\ldots\ldots\ldots\ldots\ldots\ldots(65).$$

The particular case of an indefinitely thin shell is interesting. The equation for κ may be written $f(a)=f(b)$; and, if $b=a+\delta a$, we get

$$\frac{\partial f}{\partial a}=0,$$

or

$$\frac{d}{dx}\left(\frac{\nu x-(\nu-x^2)\tan x}{\nu-x^2+\nu x\tan x}\right)=0,$$

where x is written for κa. This equation reduces to

$$x^2 \sec^2 x \{x^2 - \nu(3-\nu)\} = 0,$$

so that $$\kappa^2 a^2 = \nu(3-\nu) = \frac{4\mu(3\lambda+2\mu)}{(\lambda+2\mu)^2};$$

and the period is $$\pi a \sqrt{\left(\frac{\rho}{\mu}\frac{1-\sigma}{1+\sigma}\right)} \quad \ldots\ldots(66),$$

where $\sigma = \frac{1}{2}\lambda/(\lambda+\mu)$, is the Poisson's ratio of the material of the shell.

130. Radial Strain. Cylindrical Coordinates.

The reader will easily supply the analysis necessary to prove the following results, for cylindrical radial strain, the axes at any point being taken to be the radius, the tangent to the circular section, and the generator through the point, and the displacement being U along the radius:

(i) The strains are

$$\frac{\partial U}{\partial r}, \quad \frac{U}{r}, \quad 0, \quad 0, \quad 0, \quad 0 \quad \ldots\ldots(67).$$

(ii) The general equation of small motion is

$$(\lambda+2\mu)\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) + \rho R = \rho \frac{\partial^2 U}{\partial t^2} \quad \ldots\ldots(68).$$

(iii) The radial stress across any element of a coaxal cylinder of radius r is

$$(\lambda+2\mu)\frac{\partial U}{\partial r} + \lambda \frac{U}{r} \quad \ldots\ldots(69).$$

(iv) The solution for bodily force $R = \omega^2 r$ is

$$U = Ar + Br^{-1} - \frac{\omega^2 \rho}{8(\lambda+2\mu)} r^3 \quad \ldots\ldots(70).$$

This is sometimes taken to include the case of a circular disc[1] rotating with angular velocity ω. If the disc be complete up to the axis we must have $B = 0$, and if the edge be free

$$U = \frac{\omega^2 \rho r}{8(\lambda+2\mu)}\left(\frac{2\lambda+3\mu}{\lambda+\mu} a^2 - r^2\right) \quad \ldots\ldots(71),$$

where a is the radius of the disc. The extensions are both greatest at the axis, and there they are each equal to

$$\frac{\omega^2 \rho a^2 (2\lambda+3\mu)}{8(\lambda+\mu)(\lambda+2\mu)} \quad \ldots\ldots(72).$$

[1] A better solution of the problem of a rotating disc is given in the next article.

According to the theory of Poncelet and Saint-Venant, the cylinder will certainly tend to crack at the axis if the breaking stress T_0 of the material be less than

$$\frac{\omega^2\rho a^2\mu\,(3\lambda+2\mu)\,(2\lambda+3\mu)}{8\,(\lambda+\mu)^2\,(\lambda+2\mu)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (73),$$

and if Poisson's ratio be $\frac{1}{4}$ this condition is

$$T_0 < \tfrac{25}{96}\omega^2\rho a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74).$$

The stress in the cylinder at a distance r from the axis consists of a radial tension

$$\widehat{rr} = \frac{\omega^2\rho\,(2\lambda+3\mu)}{4\,(\lambda+2\mu)}(a^2-r^2),$$

a tension along the tangent to the circular section

$$\widehat{\theta\theta} = \frac{\omega^2\rho}{4\,(\lambda+2\mu)}[(2\lambda+3\mu)\,a^2-(2\lambda+\mu)\,r^2],$$

and a tension in the direction of the axis of the cylinder

$$\widehat{zz} = \frac{\omega^2\rho\lambda}{4\,(\lambda+2\mu)}\left[\frac{2\lambda+3\mu}{\lambda+\mu}\,a^2-2r^2\right].$$

These are principal stresses, and the maximum of each is at the axis, where $\widehat{rr}$ and $\widehat{\theta\theta}$ are $> \widehat{zz}$ and are each equal to

$$\frac{\omega^2\rho}{4\,(\lambda+2\mu)}\,(2\lambda+3\mu)\,a^2.$$

Thus Lamé's condition of safety (art. 57) would be that

$$T_0 > \omega^2\rho a^2\,\frac{2\lambda+3\mu}{4\,(\lambda+2\mu)},$$

or if Poisson's ratio be $\frac{1}{4}$,

$$T_0 > \tfrac{5}{12}\omega^2\rho a^2.$$

Thus the maximum angular velocity for safety given by Lamé's method is less than that given by Poncelet's in the ratio $\sqrt{\frac{5}{8}}$.

The maximum difference of greatest and least principal stresses is the value of $\widehat{\theta\theta}-\widehat{zz}$ at the axis, and this is

$$\frac{\omega^2\rho a^2\mu\,(2\lambda+3\mu)}{4\,(\lambda+\mu)\,(\lambda+2\mu)}.$$

On the "stress-difference" theory (art. 57) this must be less than T_0. The maximum angular velocity for safety according to this theory is $\sqrt{\frac{5}{4}}$ of that given by Poncelet's theory, Poisson's ratio being $\frac{1}{4}$.

The solution does not afford a means of experimental investigation as to the relative values of the stress-difference and the other theories, for it really refers to an infinite cylinder or a cylinder whose length is maintained constant by the requisite end tractions[1].

(v) The solution for hydrostatic pressures, p_1 inside and p_0 outside an infinite cylindrical shell of internal radius r_1 and external radius r_0, is

$$U = \frac{r_1^2 p_1 - r_0^2 p_0}{r_0^2 - r_1^2} \frac{r}{2(\lambda+\mu)} + \frac{r_0^2 r_1^2 (p_1 - p_0)}{r_0^2 - r_1^2} \frac{1}{2\mu r} \quad \ldots\ldots(75).$$

In case $p_0 = 0$ and $p_1 = p$ the greatest extension is along the circular sections of the inner cylinder, and its amount is

$$\frac{p r_1^2}{r_0^2 - r_1^2} \left[\frac{1}{2(\lambda+\mu)} + \frac{1}{2\mu} \frac{r_0^2}{r_1^2} \right] \quad \ldots\ldots(76).$$

According to the theory of Poncelet and Saint-Venant, if T_0 be the breaking stress of the material the cylinder will certainly be ruptured if

$$T_0 < \tfrac{5}{8} \frac{r_1^2 + 2r_0^2}{r_0^2 - r_1^2} p \quad \ldots\ldots(77),$$

adopting the value $\frac{1}{4}$ for Poisson's ratio. For a thin cylindrical envelope of radius r and thickness $2h$ the condition of safety is

$$\Phi p < \tfrac{32}{15} \frac{h}{r} T_0 \quad \ldots\ldots(78),$$

where Φ is the factor of safety.—This result should be compared with that in equation (58).

(vi) The solutions for purely radial vibrations of a solid cylinder of radius a is

$$U = A J_1(\kappa r) e^{ipt} \quad \ldots\ldots(79),$$

where J_1 denotes Bessel's function of order unity, and

$$\kappa^2 (\lambda + 2\mu) = p^2 \rho \quad \ldots\ldots(80).$$

We should find that κ is determined by the equation

$$\kappa a J_1'(\kappa a) + \frac{\lambda}{\lambda + 2\mu} J_1(\kappa a) = 0 \quad \ldots\ldots(81).$$

For purely radial vibrations of a cylindrical shell of radii a and b, we have in like manner

$$U = [A J_1(\kappa r) + B Y_1(\kappa r)] e^{ipt} \quad \ldots\ldots(82),$$

[1] A new solution of the problem of the rotating cylinder was communicated to the Cambridge Philosophical Society in February, 1892, by Mr Chree.

where J_1 and Y_1 denote the two kinds of Bessel's functions, and κ is determined by the equation

$$\frac{(\lambda+2\mu)\,\kappa a J_1'(\kappa a)+\lambda J_1(\kappa a)}{(\lambda+2\mu)\,\kappa a Y_1'(\kappa a)+\lambda Y_1(\kappa a)}=\frac{(\lambda+2\mu)\,\kappa b J_1'(\kappa b)+\lambda J_1(\kappa b)}{(\lambda+2\mu)\,\kappa b Y_1'(\kappa b)+\lambda Y_1(\kappa b)} \quad \ldots\ldots\ldots(83).$$

The two last problems (v) and (vi) are important in the theory of Thin Shells.

131. Strain Symmetrical about an axis. Rotating Circular Disc.

As another example[1], consider strain symmetrical about an axis.

Let the axis be the axis of z, and let r be the radius vector to any point drawn perpendicular to this axis, and θ the angle between the direction of r and a fixed plane through the axis; also let u and w be the displacements in the direction of the radius and the axis of z. Then the strains are

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial r}, \text{ the extension along } r,\\ f &= \frac{u}{r}, \text{ the extension perpendicular to the plane } (r,z),\\ g &= \frac{\partial w}{\partial z}, \text{ the extension along } z,\\ b &= \frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}, \text{ the shear of the plane } (r,z) \end{aligned}\right\} \quad \ldots(84).$$

If the material be isotropic the stresses are[2]

$$\left.\begin{aligned} \widehat{rr} &= P = \lambda\Delta + 2\mu\frac{\partial u}{\partial r},\\ \widehat{\theta\theta} &= Q = \lambda\Delta + 2\mu\frac{u}{r},\\ \widehat{zz} &= R = \lambda\Delta + 2\mu\frac{\partial w}{\partial z},\\ \widehat{rz} &= T = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(85).$$

[1] Only the leading steps of the analysis are given, and the verification is left to the reader.

[2] See art. 49.

The equations of equilibrium under "centrifugal force" $\omega^2 r$ from the axis z are

$$\left.\begin{aligned} \frac{\partial P}{\partial r}+\frac{\partial T}{\partial z}+\frac{P-Q}{r}+\omega^2\rho r&=0,\\ \frac{\partial T}{\partial r}+\frac{\partial R}{\partial z}+\frac{T}{r}&=0 \end{aligned}\right\} \quad \text{.....(86).}$$

There is no difficulty in verifying the following solution

$$\left.\begin{aligned} u&=\frac{\omega^2\rho}{8E}(1-\sigma)\{(3+\sigma)a^2r-(1+\sigma)r^3\}+\frac{\omega^2\rho}{6E}\sigma(1+\sigma)r(l^2-3z^2)\\ w&=-\frac{\omega^2\rho}{4E}\sigma\{(3+\sigma)a^2z-2(1+\sigma)r^2z\}-\frac{\omega^2\rho}{3E}\sigma^2\frac{1+\sigma}{1-\sigma}z(l^2-z^2) \end{aligned}\right\} \quad \text{.....(87),}$$

where E is the Young's modulus $\mu(3\lambda+2\mu)/(\lambda+\mu)$, and σ is the Poisson's ratio $\frac{1}{2}\lambda/(\lambda+\mu)$. It is easy to shew that this solution makes the planes $z=\pm l$ free from stress, and the cylindrical surface $r=a$ free from tangential stress, and also makes the resultant normal stress per unit length of the circumference vanish when $r=a$.

This is Mr Chree's solution[1] of the problem of the rotating circular disc. The complete solution, if it could be obtained, ought to give zero radial traction at all points of the cylindrical bounding surface, a condition which the above solution does not satisfy, *i.e.* it should make $P=0$ when $r=a$, but what it really gives is P finite when $r=a$, and

$$\int_{-l}^{l} P\,dz=0 \quad \text{.....(88),}$$

when $r=a$.

According to the principle of the equivalence of equipollent loads (p. 177), we see that for a very thin disc the solution is sufficiently accurate at all points not very near the edge. It will be found that the greatest extension is the tangential extension f_0 at the centre, $z=0$, $r=0$, and this is given by

$$Ef_0=\omega^2\rho\left[\tfrac{1}{8}(1-\sigma)(3+\sigma)a^2+\tfrac{1}{6}\sigma(1+\sigma)l^2\right] \quad \text{.....(89).}$$

This solution is quite different to that in example (iv) above. In the latter the conditions at the flat surfaces of the cylinder are altogether neglected, and it applies only to the case of an infinite cylinder rotating about its axis or of a cylinder whose length is maintained constant.

[1] *Camb. Phil. Soc. Proc.* 1890.

No solution has yet been found which satisfies all the conditions *exactly*. In this respect the problem is just as much finished and just as much unfinished as the beam-problems in the last chapter.

132. Curvilinear Distributions of Æolotropy.

In the case of an æolotropic material, with what we have called in art. 48 a curvilinear distribution of elasticity, it is convenient to refer the equations of elasticity to curvilinear coordinates, so that the directions of the axes of x_1, y_1, z_1 (art. 121) through any point are those of the axes at the point for which the energy-function takes the simplest form. The number of "elastic constants" is then the smallest possible, and those that occur are constants if the material be homogeneous. If we adopted any other mode of forming the equations the "elastic constants" of the material would vary from point to point in a manner difficult to manage. Thus in polar coordinates we may have a material which has at every point three planes of symmetry such that the axes of symmetry at any point are the directions of the meridian, the parallel, and the central radius vector at the point. As examples of curvilinear distributions we may take the problems of art. 128 and (v) of art. 130.

Taking first the cylinder-problem of art. 130, and supposing the material similar to a tetragonal crystal, whose equivalent axes of symmetry are the generator and the tangent to the circular section at any point, we shall have the energy-function W given by the equation

$$2W = A(e^2+f^2) + Cg^2 + 2Fg(e+f) + 2Hef + L(a^2+b^2) + Nc^2 \quad \ldots\ldots(90),$$

and from this the stresses are easily expressed in terms of the strains. We shall suppose the displacement purely radial and equal to U (a function of r), and thus find the strains e, f, g, a, b, c equal respectively to $0, \frac{U}{r}, \frac{dU}{dr}, 0, 0, 0$, the axes being the generator[1], the tangent to the circular section, and the radius of the cylinder through any point.

The stresses are

$$F\frac{dU}{dr} + H\frac{U}{r}, \; F\frac{dU}{dr} + A\frac{U}{r}, \; F\frac{U}{r} + C\frac{dU}{dr}, \; 0, 0, 0.$$

[1] The order is different to that in art. 125.

The equations of equilibrium under surface-tractions only reduce to

$$\frac{d}{dr}\left(F\frac{U}{r}+C\frac{dU}{dr}\right)+\frac{1}{r}\left(F\frac{U}{r}+C\frac{dU}{dr}-F\frac{dU}{dr}-A\frac{U}{r}\right)=0,$$

or

$$C\frac{d^2U}{dr^2}+\frac{C}{r}\frac{dU}{dr}-\frac{AU}{r^2}=0 \quad \text{.....(91);}$$

and the solution is

$$U=\alpha r^n+\beta r^{-n} \quad \text{.....(92),}$$

where $n^2=A/C$, and n is taken positive.

The constants are given by the equations

$$-p_0=F(\alpha r_0^{n-1}+\beta r_0^{-n-1})+Cn(\alpha r_0^{n-1}-\beta r_0^{-n-1}),$$

$$-p_1=F(\alpha r_1^{n-1}+\beta r_1^{-n-1})+Cn(\alpha r_1^{n-1}-\beta r_1^{-n-1});$$

from which we find

$$\alpha=\frac{-p_1r_0^{-n-1}+p_0r_1^{-n-1}}{(F+Cn)(r_1^{n-1}r_0^{-n-1}-r_0^{n-1}r_1^{-n-1})},$$

$$\beta=\frac{p_1r_0^{n-1}-p_0r_1^{n-1}}{(F-Cn)(r_1^{n-1}r_0^{-n-1}-r_0^{n-1}r_1^{-n-1})};$$

so that

$$U=\frac{p_1r_1^{n+1}-p_0r_0^{n+1}}{(F+Cn)(r_0^{2n}-r_1^{2n})}r^n+\frac{(r_0r_1)^{n+1}(p_1r_0^{n-1}-p_0r_1^{n-1})}{(Cn-F)(r_0^{2n}-r_1^{2n})}r^{-n} \quad \text{...(93),}$$

which agrees with equation (75) in the case of isotropy.

In Saint-Venant's solution an extension γ parallel to the axis is assumed, and γ is supposed constant; for this we may refer to Prof. Pearson's *Elastical Researches of Barré de Saint-Venant*, p. 79.

Taking next the sphere-problem of art. 128, and supposing the material of the spherical shell such that when referred to polar coordinates θ, ϕ, r the energy-function has the form (90), we find that, if the displacement be purely radial and equal to U, a function of r, the strains are

$$\frac{U}{r},\ \frac{U}{r},\ \frac{dU}{dr},\ 0,\ 0,\ 0;$$

and the stresses are

$$(A+H)\frac{U}{r}+F\frac{dU}{dr},\ (A+H)\frac{U}{r}+F\frac{dU}{dr},\ C\frac{dU}{dr}+2F\frac{U}{r};$$

and the equations of equilibrium reduce to

$$C\frac{d^2U}{dr^2}+\frac{2C}{r}\frac{dU}{dr}-2(A+H-F)\frac{U}{r^2}=0 \quad \ldots\ldots\ldots(94);$$

so that $$U=\alpha r^{n-\frac{1}{2}}+\beta r^{-n-\frac{1}{2}},$$

where $$n^2=\tfrac{1}{4}\left\{1+8\frac{A+H-F}{C}\right\},$$

and we can find, as in the cylinder-problem,

$$U=\frac{1}{r_0^{2n}-r_1^{2n}}\left\{\frac{p_1r_1^{n+\frac{3}{2}}-p_0r_0^{n+\frac{3}{2}}}{(n-\frac{1}{2})C+2F}r^{n-\frac{1}{2}}+(r_0r_1)^{2n}\frac{p_1r_1^{\frac{3}{2}-n}-p_0r_0^{\frac{3}{2}-n}}{(n+\frac{1}{2})C-2F}r^{-n-\frac{1}{2}}\right\} \quad \ldots\ldots(95),$$

which agrees with equation (53) in the case of isotropy.

The cubical dilatation of the spherical cavity is the value of $3U/r$ when $r=r_1$, and this is

$$\frac{3r_1^{n-\frac{3}{2}}}{r_0^{2n}-r_1^{2n}}\left\{\frac{p_1r_1^{n+\frac{3}{2}}-p_0r_0^{n+\frac{3}{2}}}{(n-\frac{1}{2})C+2F}+r_0^{2n}\frac{p_1r_1^{\frac{3}{2}-n}-p_0r_0^{\frac{3}{2}-n}}{(n+\frac{1}{2})C-2F}\right\} \ldots(96).$$

This result is of importance in the theory of piezometer experiments, for which a discrepancy appears to have been observed between the results obtained and the dilatation that would have place if the material were isotropic. The solution in (96) contains 3 independent constants and Saint-Venant[1] held that these could be adjusted so as to explain the experiments in question.

[1] See Pearson's *Elastical Researches of Barré de Saint-Venant*, p. 82.

CHAPTER VIII.

GENERAL SOLUTIONS.

133. Statement of the Problem.

The general problem of the Mathematical Theory of Elasticity consists in the discovery of functions u, v, w which satisfy the system of equations

$$\left.\begin{aligned} \rho X + \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \rho Y + \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} &= \rho \frac{\partial^2 v}{\partial t^2}, \\ \rho Z + \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} &= \rho \frac{\partial^2 w}{\partial t^2}, \end{aligned}\right\}$$

(where P, Q, R, S, T, U are the partial differential coefficients of a quadratic function W of the six quantities

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

) at all points within a certain closed surface, the surface of the strained solid, and also fulfil certain conditions given at the boundary.

We shall consider separately problems in which a solid is considered as held strained by the application of forces, and problems involving small motions, and shall proceed now to the consideration of the equilibrium of an isotropic solid body.

Suppose then that a mass of homogeneous isotropic elastic matter is subject to bodily forces whose components at any point

are X, Y, Z. The equations of equilibrium which hold at every point are

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u+\rho X=0,\\ (\lambda+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 v+\rho Y=0,\\ (\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w+\rho Z=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(1),$$

where Δ is the cubical dilatation given by

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In order to solve these equations we seek first any set of particular integrals in terms of X, Y, Z, and secondly the most general complementary solutions of the same equations with X, Y, Z all equal to zero. The first set of particular integrals obtained will not in general lead to values of the stresses or displacements which satisfy the boundary-conditions. In that case we have to determine the arbitrary functions or arbitrary constants, that occur in the complementary solutions, so that the complete solutions, consisting of particular integrals and complementary functions, may satisfy these conditions.

134. Formulæ for the Bodily Forces[1].

Let X, Y, Z be the components of the bodily force, per unit of mass, supposed finite continuous and one-valued functions of x, y, z throughout the body; we seek to throw X, Y, Z into the forms

$$\left.\begin{aligned}X=\frac{\partial f}{\partial x}-\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z},\\ Y=\frac{\partial f}{\partial y}-\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x},\\ Z=\frac{\partial f}{\partial z}-\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where U, V, W, and f are functions of x, y, z.

By differentiating the above equations with respect to x, y, z and adding, we obtain

$$\nabla^2 f=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=\Phi \text{ say}\ldots\ldots\ldots\ldots\ldots(4).$$

[1] The subject-matter of this and the two following articles is due to Prof. Betti—*Teoria della Elasticita. Il Nuovo Cimento*, 1872.

Let Φ' be the value of Φ at (x', y', z'), and r the distance of (x, y, z) from (x', y', z'), then a particular integral of the equation for f is the potential of a distribution whose density at (x', y', z') is $-\frac{1}{4\pi}\Phi'$, so that we may write

$$f = \psi - \frac{1}{4\pi}\iiint \frac{\Phi'}{r}\,dx'dy'dz' = \psi + F \text{ say} \quad \ldots\ldots\ldots\ldots(5),$$

where the integration extends throughout the solid, and ψ is a function which is finite continuous and one-valued within the body and satisfies the equation

$$\nabla^2\psi = 0 \quad \ldots\ldots\ldots\ldots(6);$$

we may complete the definition of ψ by subjecting it to the condition

$$\frac{\partial\psi}{\partial\nu} + \frac{\partial F}{\partial\nu} - (lX + mY + nZ) = 0 \quad \ldots\ldots\ldots\ldots(7)$$

at the boundary, (l, m, n) denoting the direction-cosines of the normal drawn outwards, and $d\nu$ the element of this normal. Thus the function f is completely determined.

Now let

$$\left.\begin{aligned} X &= \frac{\partial f}{\partial x} + G, \\ Y &= \frac{\partial f}{\partial y} + H, \\ Z &= \frac{\partial f}{\partial z} + K \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(8),$$

then G, H, K are completely determined.

By differentiating these equations with respect to x, y, z, adding, and using (4), we find

$$\frac{\partial G}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial K}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots(9);$$

and by the condition (7) we have

$$lG + mH + nK = 0 \quad \ldots\ldots\ldots\ldots(10)$$

at the boundary.

Let

$$A = -\frac{1}{4\pi}\iiint \frac{G'dx'dy'dz'}{r} \quad \ldots\ldots\ldots\ldots(11),$$

where, as in the case of Φ', G' is the value of G at (x', y', z'), and in like manner let

$$\left.\begin{aligned} B &= -\frac{1}{4\pi}\iiint \frac{H'dx'dy'dz'}{r}, \\ C &= -\frac{1}{4\pi}\iiint \frac{K'dx'dy'dz'}{r} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(12);$$

then we have

$$\nabla^2 A = G,\ \nabla^2 B = H,\ \nabla^2 C = K.$$

From the definitions of A, B, C we obtain

$$\begin{aligned}\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}+\frac{\partial C}{\partial z} &= -\frac{1}{4\pi}\iiint\left(G'\frac{\partial}{\partial x}+H'\frac{\partial}{\partial y}+K'\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)dx'dy'dz' \\ &= \frac{1}{4\pi}\iiint\left(G'\frac{\partial}{\partial x'}+H'\frac{\partial}{\partial y'}+K'\frac{\partial}{\partial z'}\right)\left(\frac{1}{r}\right)dx'dy'dz' \\ &= \frac{1}{4\pi}\iint(lG'+mH'+nK')\frac{1}{r}\,dS \\ &\quad -\frac{1}{4\pi}\iiint\left(\frac{\partial G'}{\partial x'}+\frac{\partial H'}{\partial y'}+\frac{\partial K'}{\partial z'}\right)\frac{1}{r}\,dx'dy'dz' \\ &= 0 \text{ identically.}\end{aligned}$$

We can now write

$$G = \nabla^2 A = \frac{\partial}{\partial y}\left(\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}\right)-\frac{\partial}{\partial z}\left(\frac{\partial C}{\partial x}-\frac{\partial A}{\partial z}\right) \quad \text{.....(13),}$$

and we have similar equations for H and K.

Hence, if

$$-U=\frac{\partial B}{\partial z}-\frac{\partial C}{\partial y},\ -V=\frac{\partial C}{\partial x}-\frac{\partial A}{\partial z},\ -W=\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x} \quad \text{...(14),}$$

X, Y, Z will be thrown into the forms (3), and all the functions f, U, V, W will be well defined.

135. Interpretation.

Consider any surface σ drawn within the body. The surface-integral of the normal component of the system of forces depending on f is $\rho\iiint \Phi dT$, where dT is the element of the volume within the surface σ, and, when the surface is contracted to a point, we see that this system of forces tends to vary the volume of an element.

The surface-integral of the normal component of the G, H, K system is

$$\rho\iint(lG+mH+nK)\,d\sigma = \rho\iiint\left(\frac{\partial G}{\partial x}+\frac{\partial H}{\partial y}+\frac{\partial K}{\partial z}\right)dT = 0,$$

so that this system does not tend to alter the volume of an element.

Consider the line-integral of the tangential component of this latter system along any closed line s, and let dS be an element of a surface having the line s for an edge, then this line-integral is

$$\int Gdx + Hdy + Kdz,$$

and, by the theorem for the transformation of line-integrals and surface-integrals, this is

$$\iint \left\{ l\left(\frac{\partial K}{\partial y} - \frac{\partial H}{\partial z}\right) + m\left(\frac{\partial G}{\partial z} - \frac{\partial K}{\partial x}\right) + n\left(\frac{\partial H}{\partial x} - \frac{\partial G}{\partial y}\right) \right\} dS.$$

Thus if s be a very small closed curve in the plane (y, z), and S_1 its area, the line-integral in question is $S_1 \nabla^2 U$, so that the system G, H, K tends to produce rotation of the elements.

136. Particular Integrals for the Bodily Forces.

Now let u, v, w be the displacements at any point of the body, and suppose u, v, w expressed in the same way as X, Y, Z in the forms

$$\left.\begin{aligned} u &= \frac{\partial \phi}{\partial x} - \frac{\partial N}{\partial y} + \frac{\partial M}{\partial z}, \\ v &= \frac{\partial \phi}{\partial y} - \frac{\partial L}{\partial z} + \frac{\partial N}{\partial x}, \\ w &= \frac{\partial \phi}{\partial z} - \frac{\partial M}{\partial x} + \frac{\partial L}{\partial y} \end{aligned}\right\} \quad \text{.....(15).}$$

Then $$\Delta = \nabla^2 \phi,$$

and $$\nabla^2 u = \frac{\partial \Delta}{\partial x} - \frac{\partial \nabla^2 N}{\partial y} + \frac{\partial \nabla^2 M}{\partial z}.$$

The equations of equilibrium become three such as

$$(\lambda + 2\mu)\frac{\partial \nabla^2 \phi}{\partial x} + \mu\left(-\frac{\partial \nabla^2 N}{\partial y} + \frac{\partial \nabla^2 M}{\partial z}\right) + \rho\left(\frac{\partial f}{\partial x} - \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) = 0.$$

Hence we have a solution in the form

$$\left.\begin{aligned} \phi &= \frac{\rho}{4\pi(\lambda + 2\mu)} \iiint \frac{f'}{r}\, dx'dy'dz', \\ L &= \frac{\rho}{4\pi\mu} \iiint \frac{U'}{r}\, dx'dy'dz' \end{aligned}\right\} \quad \text{.....(16),}$$

and similar forms for M and N, where as before f', U' are the

values of f, U at (x', y', z'). Hence we can write down u in the form

$$u = \frac{\rho}{4\pi(\lambda+2\mu)} \iiint \frac{f}{r^2} \cos \widehat{rx}\, dx'dy'dz'$$

$$- \frac{\rho}{4\pi\mu} \iiint \left\{ \frac{W'}{r^2} \cos \widehat{ry} - \frac{V'}{r^2} \cos \widehat{rz} \right\} dx'dy'dz' \ldots\ldots (17),$$

where $\cos \widehat{rx}$ is the cosine of the angle between the axis x and the line r drawn from (x, y, z) to (x', y', z'), and v and w can be written down by symmetry. These values of u, v, w are particular integrals of the equations of equilibrium. They will not however in general satisfy the boundary-conditions.

We notice that in accordance with our interpretation of f, U, V, W the cubical dilatation is $-\rho f/(\lambda+2\mu)$.

137. Second form of Particular Integral.

Another method of obtaining the particular integral will be given later (ch. IX. art. 150), where we shall shew that, if X', Y', Z' be the bodily forces, per unit mass, applied at the point (x', y', z'), the equations of equilibrium can be satisfied by the forms

$$\left.\begin{aligned} u &= \frac{\rho}{4\pi\mu} \iiint \left[\frac{X'}{r} - \frac{\lambda+\mu}{2(\lambda+2\mu)} \frac{\partial}{\partial x} \left(X' \frac{\partial r}{\partial x} + Y' \frac{\partial r}{\partial y} + Z' \frac{\partial r}{\partial z} \right) \right] dx'dy'dz', \\ v &= \frac{\rho}{4\pi\mu} \iiint \left[\frac{Y'}{r} - \frac{\lambda+\mu}{2(\lambda+2\mu)} \frac{\partial}{\partial y} \left(X' \frac{\partial r}{\partial x} + Y' \frac{\partial r}{\partial y} + Z' \frac{\partial r}{\partial z} \right) \right] dx'dy'dz', \\ w &= \frac{\rho}{4\pi\mu} \iiint \left[\frac{Z'}{r} - \frac{\lambda+\mu}{2(\lambda+2\mu)} \frac{\partial}{\partial z} \left(X' \frac{\partial r}{\partial x} + Y' \frac{\partial r}{\partial y} + Z' \frac{\partial r}{\partial z} \right) \right] dx'dy'dz' \end{aligned} \right\}$$

$$\ldots\ldots (18).$$

Solutions equivalent to these are given in Thomson and Tait's *Natural Philosophy*, Part II. art. 731.

138. Particular Integral for Forced Vibrations.

Suppose the solid executes forced vibrations, under the action of periodic forces. Then we have to take X, Y, Z and consequently f, U, V, W all proportional to e^{ipt}, where $2\pi/p$ is the period.

In the forced vibrations u, v, w will also be proportional to e^{ipt}, and thus the equations of small vibration may be written in such forms as

$$(\lambda+\mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u + \rho p^2 u + \rho \left(\frac{\partial f}{\partial x} - \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} \right) = 0 \ldots (19).$$

Now substituting from (15), and writing

$$h^2 = \rho p^2/(\lambda + 2\mu), \quad \kappa^2 = \rho p^2/\mu \ldots\ldots\ldots\ldots\ldots (20),$$

we have three such equations as

$$(\lambda + 2\mu)\frac{\partial}{\partial x}\left[(\nabla^2 + h^2)\,\phi + \frac{\rho}{\lambda + 2\mu} f\right] - \mu\frac{\partial}{\partial y}\left[(\nabla^2 + \kappa^2)\,N + \frac{\rho}{\mu} W\right]$$
$$+ \mu\frac{\partial}{\partial z}\left[(\nabla^2 + \kappa^2)\,M + \frac{\rho}{\mu} V\right] = 0 \ldots\ldots\ldots (21),$$

and thus all the equations can be satisfied by making ϕ a solution of

$$(\nabla^2 + h^2)\,\phi + \frac{\rho}{\lambda + 2\mu} f = 0 \ldots\ldots\ldots\ldots\ldots (22),$$

and L, M, N solutions of such equations as

$$(\nabla^2 + \kappa^2)\,L + \frac{\rho}{\mu} U = 0 \ldots\ldots\ldots\ldots\ldots (23).$$

Now we know that a particular solution of (22) is

$$\phi = \frac{\rho}{4\pi(\lambda + 2\mu)} \iiint \frac{f' e^{-ihr}}{r}\, dx'dy'dz' \ldots\ldots\ldots\ldots (24)$$

(see Lord Rayleigh's *Theory of Sound*, vol. II. art. 277), and in like manner for L, M, N we have such solutions as

$$L = \frac{\rho}{4\pi\mu} \iiint \frac{U' e^{-i\kappa r}}{r}\, dx'dy'dz' \ldots\ldots\ldots\ldots\ldots (25).$$

The values of u, v, w hence obtained are particular integrals of the equations of small motion (21), but they do not in general satisfy the boundary-conditions.

139. Particular Class of Cases.

When the bodily forces have a potential f which satisfies Laplace's equation, these particular solutions are very much simplified.

For equilibrium we may take

$$u\,dx + v\,dy + w\,dz = d\phi \ldots\ldots\ldots\ldots\ldots (26).$$

Then $\Delta = \nabla^2\phi$, and we have three such equations as

$$\frac{\partial}{\partial x}\{(\lambda + 2\mu)\,\nabla^2\phi + \rho f\} = 0,$$

whence we may take

$$\nabla^2\phi + \frac{\rho}{\lambda + 2\mu} f = 0 \ldots\ldots\ldots\ldots\ldots (27).$$

Now f may be thrown into the form

$$f = r\frac{\partial F}{\partial r} + \tfrac{3}{2}F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

where $r^2 = x^2 + y^2 + z^2$, and F satisfies Laplace's equation, and then

$$\nabla^2(\tfrac{1}{4}r^2F) = r\frac{\partial F}{\partial r} + \tfrac{3}{2}F = f.$$

Hence
$$\phi = -\tfrac{1}{4}r^2\frac{\rho}{\lambda + 2\mu}F \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29),$$

and
$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}, \quad w = \frac{\partial\phi}{\partial z}$$

constitute a set of particular integrals.

For forced vibrations, taking the equations such as

$$(\lambda + \mu)\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u + \rho p^2 u + \rho\frac{\partial f}{\partial x} = 0 \ldots\ldots\ldots\ldots(30),$$

where f satisfies $\nabla^2 f = 0$, and has the time-factor e^{ipt}, we may put

$$u = -\frac{1}{p^2}\frac{\partial f}{\partial x}, \quad v = -\frac{1}{p^2}\frac{\partial f}{\partial y}, \quad w = -\frac{1}{p^2}\frac{\partial f}{\partial z} \ldots\ldots\ldots(31),$$

then these make

$$\nabla^2 u = 0, \quad \nabla^2 v = 0, \quad \nabla^2 w = 0, \quad \Delta = 0,$$

and we have a set of particular integrals.

140. Description of Betti's Method of Integration.

Prof. Betti has developed, by the aid of his theorem (art. 68), a general method of integrating the equations of elasticity, for an isotropic solid of any shape, with any given boundary-conditions, when the problem can be solved for the same solid with a certain set of boundary-conditions. In this method we seek in the first place to determine the cubical dilatation and the three component rotations, and from these we find the corresponding displacements. We have already shewn that it is always possible to find a particular integral for the bodily forces; so that we may divide the problem into two parts: (1) the determination of a system of particular displacements which satisfy the equations containing the bodily forces but do not satisfy the boundary-conditions; (2) the determination of a system of displacements which satisfy the equations when the bodily forces are null and which also satisfy

arbitrary boundary-conditions. It is with the latter problem that we shall here occupy ourselves.

We have to find a solution of the equations

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u=0,\\ (\lambda+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 v=0,\\ (\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(32),$$

which hold at all points of the solid.

We shall consider first the problem of determining the cubical dilatation Δ and the three rotations $\varpi_1, \varpi_2, \varpi_3$, so as to satisfy the differential equations, and so that it may be possible to satisfy the boundary-conditions; and we shall suppose that at the boundary of the solid either the surface-tractions or the displacements are given functions. When Δ, ϖ_1, ϖ_2, ϖ_3 are known, we have

$$\left.\begin{aligned}\nabla^2 u=-\frac{\lambda+\mu}{\mu}\frac{\partial\Delta}{\partial x},\\ \nabla^2 v=-\frac{\lambda+\mu}{\mu}\frac{\partial\Delta}{\partial y},\\ \nabla^2 w=-\frac{\lambda+\mu}{\mu}\frac{\partial\Delta}{\partial z}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Hence, if the surface-displacements be given, we have to find u, v, w to satisfy equations of the form $\nabla^2 u =$ a given function of x, y, z, and $u =$ a given function at the boundary.

If the surface-tractions F, G, H be given the boundary-conditions can be written, by (15) of art. 29, in the forms

$$\left.\begin{aligned}\frac{\partial u}{\partial\nu}=\frac{F}{2\mu}-l\frac{\lambda}{2\mu}\Delta-m\varpi_3+n\varpi_2,\\ \frac{\partial v}{\partial\nu}=\frac{G}{2\mu}+l\varpi_3-m\frac{\lambda}{2\mu}\Delta-n\varpi_1,\\ \frac{\partial w}{\partial\nu}=\frac{H}{2\mu}-l\varpi_2+m\varpi_1-n\frac{\lambda}{2\mu}\Delta\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(34),$$

where (l, m, n) are the direction-cosines of the normal $(d\nu)$ to the boundary drawn outwards from the space occupied by the solid. Thus we have to find u, v, w to satisfy equations of the form

$\nabla^2 u =$ a given function of (x, y, z), and $\dfrac{\partial u}{\partial \nu} =$ a given function at the boundary.

Now Prof. Betti has shewn that we can find the value of Δ, at any point (x', y', z'), so that the surface-displacements may be given functions, if we can find systems of displacements ξ, η, ζ which become equal at any point (x, y, z) of the surface to $-\dfrac{\partial r^{-1}}{\partial x}, -\dfrac{\partial r^{-1}}{\partial y}, -\dfrac{\partial r^{-1}}{\partial z}$, where r is the distance between the points (x, y, z) and (x', y', z'); and we can find Δ so that the surface-tractions may be given functions if we can find displacements (ξ, η, ζ) such that the surface-tractions that would produce them are those that would occur if near the surface the displacements were $-\dfrac{\partial r^{-1}}{\partial x}, -\dfrac{\partial r^{-1}}{\partial y}, -\dfrac{\partial r^{-1}}{\partial z}$; and he has given similar methods for the determination of $\varpi_1, \varpi_2, \varpi_3$.

141. Determination of the Cubical Dilatation.

Consider first the system of displacements

$$u_0 = \frac{\partial r^{-1}}{\partial x} + \xi_0, \quad v_0 = \frac{\partial r^{-1}}{\partial y} + \eta_0, \quad w_0 = \frac{\partial r^{-1}}{\partial z} + \zeta_0 \ldots\ldots (35),$$

where r is the distance of any point (x, y, z) from a particular point (x', y', z') of the solid, and ξ_0, η_0, ζ_0 are finite, continuous, and one-valued throughout the volume V enclosed by the surface S of the solid. We shall shew that, if ξ_0, η_0, ζ_0 be suitably determined, we can hence obtain the value of Δ. The quantities

$$\frac{\partial r^{-1}}{\partial x}, \quad \frac{\partial r^{-1}}{\partial y}, \quad \frac{\partial r^{-1}}{\partial z} \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

satisfy the equations of equilibrium (32) at all points which lie within the volume V', enclosed between the surface S and any small closed surface S' surrounding the point (x', y', z'). Hence if ξ_0, η_0, ζ_0 satisfy these equations throughout the volume V, the displacements u_0, v_0, w_0 given by (4) will satisfy the equations throughout the volume V'. Let $F_0 + L_0, G_0 + M_0, H_0 + N_0$ be the surface-tractions on S arising from the displacements u_0, v_0, w_0, and suppose L_0, M_0, N_0 are the parts contributed to these surface-tractions by the displacements ξ_0, η_0, ζ_0. Let $F_0' + L_0', G_0' + M_0', H_0' + N_0'$ be the surface-tractions on S' arising from the same set

of displacements, and L_0', M_0', N_0' the parts contributed by ξ_0, η_0, ζ_0.

Let u, v, w be any system of displacements finite, continuous, and one-valued throughout V, and requiring no bodily force for its maintenance, F, G, H the resulting surface-tractions on S, F', G', H' the resulting surface-tractions on S'. Let us apply Prof. Betti's reciprocal theorem (art. 68) to the systems (u, v, w) and (u_0, v_0, w_0) and the space V' between the surfaces S and S'; then, since there is no bodily force, we have

$$\iint (Fu_0 + Gv_0 + Hw_0)\, dS + \iint (F'u_0 + G'v_0 + H'w_0)\, dS'$$
$$= \iint \{(F_0 + L_0)\, u + (G_0 + M_0)\, v + (H_0 + N_0)\, w\}\, dS$$
$$+ \iint \{(F_0' + L_0')\, u + (G_0' + M_0')\, v + (H_0' + N_0')\, w\}\, dS' \ldots\ldots\ldots (37).$$

We shall find the limiting form of this equation when S' is contracted to a point.

The left-hand side is

$$\iint \left[F\left(\frac{\partial r^{-1}}{\partial x} + \xi_0\right) + G\left(\frac{\partial r^{-1}}{\partial y} + \eta_0\right) + H\left(\frac{\partial r^{-1}}{\partial z} + \zeta_0\right)\right] dS$$
$$+ \iint \left(F' \frac{\partial r^{-1}}{\partial x} + G' \frac{\partial r^{-1}}{\partial y} + H' \frac{\partial r^{-1}}{\partial z}\right) dS',$$

and the right-hand side is

$$\iint [(F_0 + L_0)\, u + (G_0 + M_0)\, v + (H_0 + N_0)\, w]\, dS$$
$$+ \iint (F_0'u + G_0'v + H_0'w)\, dS',$$

since the integrals $\iint (F'\xi_0 + G'\eta_0 + H'\zeta_0)\, dS'$

and $\iint (L_0'u + M_0'v + N_0'w)\, dS'$

vanish when S' is contracted to a point, the functions to be integrated being finite.

To calculate $\iint \left\{F' \frac{\partial r^{-1}}{\partial x} + G' \frac{\partial r^{-1}}{\partial y} + H' \frac{\partial r^{-1}}{\partial z}\right\} dS'$

we may take the origin at (x', y', z'), and the surface S' a sphere of small radius, whose centre is the origin. Then, remembering that the normal to S' must be drawn towards its centre, we have, by (15) of art. 29,

$$F' \frac{\partial r^{-1}}{\partial x} + G' \frac{\partial r^{-1}}{\partial y} + H' \frac{\partial r^{-1}}{\partial z}$$
$$= \frac{x}{r^3}\left[\lambda\Delta \frac{x}{r} + 2\mu \frac{\partial u}{\partial r} + \mu \frac{y}{r}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \mu \frac{z}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\right]$$
$$+ \text{two similar expressions},$$
$$= \frac{\lambda\Delta}{r^2} + \frac{2\mu}{r^2}\left(\frac{x}{r}\frac{\partial u}{\partial r} + \frac{y}{r}\frac{\partial v}{\partial r} + \frac{z}{r}\frac{\partial w}{\partial r}\right).$$

Again

$$F_0'u + G_0'v + H_0'w = -2\mu\left[u\frac{\partial}{\partial r}\left(\frac{\partial r^{-1}}{\partial x}\right) + v\frac{\partial}{\partial r}\left(\frac{\partial r^{-1}}{\partial y}\right) + w\frac{\partial}{\partial r}\left(\frac{\partial r^{-1}}{\partial z}\right)\right]$$

$$= -4\mu\frac{ux+vy+wz}{r^4}.$$

Thus equation (37) becomes

$$\iint\left[\frac{\lambda\Delta}{r^2} + \frac{2\mu}{r^3}\left\{x\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) + y\left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right) + z\left(\frac{\partial w}{\partial r} + \frac{2w}{r}\right)\right\}\right] dS'$$

$$= -\iint\left[F\left(\frac{\partial r^{-1}}{\partial x} + \xi_0\right) + G\left(\frac{\partial r^{-1}}{\partial y} + \eta_0\right) + H\left(\frac{\partial r^{-1}}{\partial z} + \zeta_0\right)\right] dS$$

$$+ \iint[(F_0 + L_0)u + (G_0 + M_0)v + (H_0 + N_0)w]\, dS \ldots\ldots\ldots(38).$$

Now $$xr\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = \frac{\partial}{\partial r}(uxr),$$

and $$\iiint\frac{\partial u}{\partial x}\,dx\,dy\,dz = \iint\frac{ux}{r}\,dS' = \iint uxr\,d\omega,$$

if $r^2 d\omega = dS'$.

Hence $$\int_0^r r^2 dr \iint\frac{\partial u}{\partial x}\,d\omega = \iint uxr\,d\omega;$$

and therefore, differentiating,

$$r^2\iint\frac{\partial u}{\partial x}\,d\omega = \iint xr\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right)d\omega;$$

so that $$\iint\frac{1}{r^2}\frac{\partial u}{\partial x}\,dS' = \iint\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right)\frac{x}{r^3}\,dS'.$$

Thus equation (38) is transformed into

$$4\pi(\lambda+2\mu)\Delta = -\iint\left[F\left(\frac{\partial r^{-1}}{\partial x} + \xi_0\right) + G\left(\frac{\partial r^{-1}}{\partial y} + \eta_0\right) + H\left(\frac{\partial r^{-1}}{\partial z} + \zeta_0\right)\right] dS$$

$$+ \iint[(F_0 + L_0)u + (G_0 + M_0)v + (H_0 + N_0)w]\, dS \ldots\ldots(39).$$

This gives the value of Δ at (x', y', z'), when the surface-tractions are F, G, H, and the surface-displacements are u, v, w.

If the surface-displacements be given, then supposing we can find ξ_0, η_0, ζ_0 so as to satisfy the equations of equilibrium, and so as to make

$$\xi_0 = -\frac{\partial r^{-1}}{\partial x}, \quad \eta_0 = -\frac{\partial r^{-1}}{\partial y}, \quad \zeta_0 = -\frac{\partial r^{-1}}{\partial z}$$

at the surface, r being the distance of any point on the surface from (x', y', z'), we shall have to calculate thence the sets of

surface-tractions F_0, G_0, H_0 and L_0, M_0, N_0. When this is done the value of Δ at (x', y', z') can be expressed in the form

$$\Delta = \frac{1}{4\pi(\lambda+2\mu)} \iint [(F_0 + L_0)u + (G_0 + M_0)v + (H_0 + N_0)w]\, dS \ldots (40),$$

where u, v, w are the given surface-displacements.

If the surface-tractions be given, we first calculate the tractions F_0, G_0, H_0 as if $\frac{\partial r^{-1}}{\partial x}$, $\frac{\partial r^{-1}}{\partial y}$, $\frac{\partial r^{-1}}{\partial z}$ were the displacements; then we find ξ_0, η_0, ζ_0 a system of displacements which satisfy the general equations of equilibrium and the particular boundary-conditions $F = -F_0$, $G = -G_0$, $H = -H_0$, *i.e.* we make $F_0 + L_0$, $G_0 + M_0$, $H_0 + N_0$ vanish. When this is done the value of Δ at (x', y', z') can be expressed in the form

$$\Delta = -\frac{1}{4\pi(\lambda+2\mu)} \iint \left[F\left(\frac{\partial r^{-1}}{\partial x} + \xi_0\right) + G\left(\frac{\partial r^{-1}}{\partial y} + \eta_0\right) + H\left(\frac{\partial r^{-1}}{\partial z} + \zeta_0\right) \right] dS \ldots\ldots (41),$$

where F, G, H are the given surface-tractions.

142. Determination of the Rotations.

To determine the rotation $\varpi_3 = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$, we take

$$u_3 = \frac{\partial r^{-1}}{\partial y} + \xi_3, \quad v_3 = -\frac{\partial r^{-1}}{\partial x} + \eta_3, \quad w_3 = \zeta_3 \ldots\ldots (42),$$

where r has the same meaning as before, and ξ_3, η_3, ζ_3 are finite, continuous, and one-valued throughout the solid, and are a possible system of displacements satisfying the differential equations of equilibrium. Then we form the surface-tractions $F_3 + L_3$, $G_3 + M_3$, $H_3 + N_3$ on S, where L_3, M_3, N_3 are the parts contributed by the displacements ξ_3, η_3, ζ_3, and the similar set $F_3' + L_3'$, $G_3' + M_3'$, $H_3' + N_3'$ on S', and take any other set of displacements u, v, w, and the corresponding surface-tractions F, G, H on S and F', G', H' on S', and apply Prof. Betti's theorem as before to the volume between S and S' when S' is contracted to a point. We thus obtain the equation

$$\begin{aligned} \iint \left[F\left(\frac{\partial r^{-1}}{\partial y} + \xi_3\right) + G\left(-\frac{\partial r^{-1}}{\partial x} + \eta_3\right) + H\zeta_3 \right] dS \\ + \iint \left(F' \frac{\partial r^{-1}}{\partial y} - G' \frac{\partial r^{-1}}{\partial x} \right) dS' \\ = \iint [(F_3 + L_3)u + (G_3 + M_3)v + (H_3 + N_3)w]\, dS \\ + \iint (F_3'u + G_3'v + H_3'w)\, dS' \ldots\ldots (43). \end{aligned}$$

As before, take (x', y', z') as origin, and S'' a small sphere described round this point as centre, then, by (15) of art. 29,

$$F'\frac{\partial r^{-1}}{\partial y} - G'\frac{\partial r^{-1}}{\partial x} = 2\mu\left(\frac{y}{r^3}\frac{\partial u}{\partial r} - \frac{x}{r^3}\frac{\partial v}{\partial r}\right) + \frac{\mu}{r^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$- \mu\frac{z}{r^3}\left[\frac{x}{r}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \frac{y}{r}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{z}{r}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right],$$

and

$$F_3'u + G_3'v + H_3'w = \mu u\left[2\frac{\partial}{\partial r}\left(\frac{y}{r^3}\right) + \frac{y}{r}\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) - \frac{z}{r}\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right]$$
$$+ \mu v\left[-2\frac{\partial}{\partial r}\left(\frac{x}{r^3}\right) + \frac{z}{r}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \frac{x}{r}\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$
$$+ \mu w\left[\frac{x}{r}\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) - \frac{y}{r}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right)\right].$$

Hence $$F'\frac{\partial r^{-1}}{\partial y} - G'\frac{\partial r^{-1}}{\partial x} - (F_3'u + G_3'v + H_3'w)$$

$$= 2\mu\left[\left\{\frac{y}{r^3}\frac{\partial u}{\partial r} - u\frac{\partial}{\partial r}\left(\frac{y}{r^3}\right)\right\} - \left\{\frac{x}{r^3}\frac{\partial v}{\partial r} - v\frac{\partial}{\partial r}\left(\frac{x}{r^3}\right)\right\}\right] + \frac{\mu}{r^3}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$+ \frac{\mu}{r^3}\left[\frac{x}{r}\left\{\frac{\partial(vz)}{\partial z} - \frac{\partial(wz)}{\partial y}\right\} + \frac{y}{r}\left\{\frac{\partial(wz)}{\partial x} - \frac{\partial(uz)}{\partial z}\right\} + \frac{z}{r}\left\{\frac{\partial(uz)}{\partial y} - \frac{\partial(vz)}{\partial x}\right\}\right].$$

The integral of the last line over S'' vanishes identically.

The first line is

$$\frac{2\mu}{r^4}\left\{\frac{\partial(ury)}{\partial r} - \frac{\partial(vrx)}{\partial r}\right\} + \frac{\mu}{r^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right);$$

and, working as before, we find for the surface-integral the value of

$$-4\pi\mu\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

at (x', y', z').

Hence at (x', y', z') we have

$$\varpi_3 = \frac{1}{8\pi\mu}\iint\left[F\left(\frac{\partial r^{-1}}{\partial y} + \xi_3\right) + G\left(-\frac{\partial r^{-1}}{\partial x} + \eta_3\right) + H\zeta_3\right]dS$$
$$- \frac{1}{8\pi\mu}\iint\left[(F_3 + L_3)u + (G_3 + M_3)v + (H_3 + N_3)w\right]dS \ldots (44).$$

If the surface-displacements be given, we have to find ξ_3, η_3, ζ_3 a system of displacements to satisfy the general equations of equilibrium and to make

$$\xi_3 = -\frac{\partial r^{-1}}{\partial y}, \quad \eta_3 = \frac{\partial r^{-1}}{\partial x}, \quad \zeta_3 = 0 \ldots\ldots\ldots\ldots (45)$$

at the surface ; then we calculate two sets of surface-tractions, viz.: L_3, M_3, N_3 corresponding to ξ_3, η_3, ζ_3 and F_3, G_3, H_3 corresponding to displacements $\frac{\partial r^{-1}}{\partial y}$, $-\frac{\partial r^{-1}}{\partial x}$, 0.

When this is done the rotation ϖ_3 can be expressed in the form

$$\varpi_3 = -\frac{1}{8\pi\mu}\iint[(F_3+L_3)u+(G_3+M_3)v+(H_3+N_3)w]\,dS \ldots (46),$$

where u, v, w are the given surface-displacements.

If the surface-tractions be given, we have to find ξ_3, η_3, ζ_3 a system of displacements to satisfy the general equations of equilibrium, and to make the surface-tractions

$$F=-F_3 \quad G=-G_3 \quad H=-H_3 \ldots\ldots\ldots\ldots (47),$$

where F_3, G_3, H_3 are calculated as if the displacements were $\frac{\partial r^{-1}}{\partial y}$, $-\frac{\partial r^{-1}}{\partial x}$, 0 ; then the rotation ϖ_3 can be expressed in the form

$$\varpi_3 = \frac{1}{8\pi\mu}\iint\left[F\left(\frac{\partial r^{-1}}{\partial y}+\xi_3\right)+G\left(-\frac{\partial r^{-1}}{\partial x}+\eta_3\right)+H\zeta_3\right]dS \ldots (48),$$

where F, G, H are the given surface-tractions.

In like manner ϖ_1, ϖ_2 can be determined.

To find ϖ_1, when the surface-tractions are given, we seek a system of displacements ξ_1, η_1, ζ_1 which satisfy the equations of equilibrium, and which would be produced by surface-tractions equal to those that would act at the surface if the displacements near the surface were 0, $-\frac{\partial r^{-1}}{\partial z}$, $\frac{\partial r^{-1}}{\partial y}$, then

$$8\pi\mu\varpi_1 = \iint\left[F\xi_1+G\left(\frac{\partial r^{-1}}{\partial z}+\eta_1\right)+H\left(-\frac{\partial r^{-1}}{\partial y}+\zeta_1\right)\right]dS \ldots (49),$$

where F, G, H are the given surface-tractions.

To find ϖ_2 we seek a system of displacements ξ_2, η_2, ζ_2 which satisfy the equations of equilibrium, and would be produced by surface-tractions equal to those that would act at the surface if the displacements near the surface were $\frac{\partial r^{-1}}{\partial z}$, 0, $-\frac{\partial r^{-1}}{\partial x}$, then

$$8\pi\mu\varpi_2 = \iint\left[F\left(-\frac{\partial r^{-1}}{\partial z}+\xi_2\right)+G\eta_2+H\left(\frac{\partial r^{-1}}{\partial x}+\zeta_2\right)\right]dS \ldots (50).$$

We might state in similar language the methods of determining ϖ_1 and ϖ_2 when the surface-displacements are given, but this case is of less importance as u, v, w can be determined when Δ is known without the previous determination of ϖ_1, ϖ_2, ϖ_3.

Prof. Betti has applied his method to develope the solutions of problems concerned with spherical boundaries, and has obtained results in terms of definite integrals extended over the bounding surfaces. Similar results were found by Borchardt using a different analysis. (See Introduction.) The same method has been applied by Signor Cerruti to determine the state of strain in the interior of a solid bounded by an infinite plane at which given conditions are satisfied. We shall consider this problem in the following chapter.

CHAPTER IX.

THE PROBLEM OF BOUSSINESQ AND CERRUTI.

DISPLACEMENT IN A SOLID BOUNDED BY AN INFINITE PLANE—SURFACE-TRACTIONS GIVEN.

143. Statement of the Problem.

Suppose a solid bounded on one side by an infinite plane, and otherwise unlimited. If the points of the plane be made to execute given displacements, or if given tractions be applied to the plane, strains will be produced in the interior. The problem of determining the displacements produced was first attempted by Lamé and Clapeyron and was afterwards solved by M. Boussinesq[1] and Signor Valentino Cerruti[2]. We shall give Signor Cerruti's solution, and shall investigate particular cases by the method of M. Boussinesq. We begin with the case where the surface-displacements are given.

144. Determination of the dilatation.

Suppose the solid is bounded by the plane $z = 0$, and that the displacements u, v, w are given functions of x, y when $z = 0$. We have in the first place to determine Δ at any point (x', y', z') of the solid. For this purpose we require a system of displacements ξ_0, η_0, ζ_0, which satisfy the equations of equilibrium, and, at the surface, are equal to

$$-\frac{\partial r^{-1}}{\partial x}, \quad -\frac{\partial r^{-1}}{\partial y}, \quad -\frac{\partial r^{-1}}{\partial z},$$

r being the distance between (x, y, z) and (x', y', z').

Let (x_1, y_1, z_1) be the image of (x', y', z') in the plane $z = 0$, so that

$$x_1 = x', \ y_1 = y', \ z_1 = -z' \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and let

$$R^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \ldots\ldots\ldots\ldots\ldots (2);$$

[1] *Applications des Potentiels, directes, inverses, logarithmiques.* Paris, 1885.

[2] 'Ricerche intorno all' equilibrio de corpi elastici isotropi'—*Reale Accademia dei Lincei*, Rome, 1882.

then we have, when $z = 0$,

$$\frac{\partial r^{-1}}{\partial x} = \frac{\partial R^{-1}}{\partial x}, \quad \frac{\partial r^{-1}}{\partial y} = \frac{\partial R^{-1}}{\partial y}, \quad \frac{\partial r^{-1}}{\partial z} = -\frac{\partial R^{-1}}{\partial z}.$$

Thus $-\partial R^{-1}/\partial x$, $-\partial R^{-1}/\partial y$, $\partial R^{-1}/\partial z$ are functions which satisfy the boundary-conditions, but they do not satisfy the differential equations of equilibrium. We therefore take

$$\xi_0 = \xi' - \frac{\partial R^{-1}}{\partial x}, \quad \eta_0 = \eta' - \frac{\partial R^{-1}}{\partial y}, \quad \zeta_0 = \zeta' + \frac{\partial R^{-1}}{\partial z},$$

where ξ', η', ζ' vanish with z. Now if these be a system of displacements the differential equations of equilibrium become

$$\left.\begin{aligned}
(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial \xi'}{\partial x}+\frac{\partial \eta'}{\partial y}+\frac{\partial \zeta'}{\partial z}+2\frac{\partial^2 R^{-1}}{\partial z^2}\right)+\mu\nabla^2\xi' = 0,\\
(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial \xi'}{\partial x}+\frac{\partial \eta'}{\partial y}+\frac{\partial \zeta'}{\partial z}+2\frac{\partial^2 R^{-1}}{\partial z^2}\right)+\mu\nabla^2\eta' = 0,\\
(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial \xi'}{\partial x}+\frac{\partial \eta'}{\partial y}+\frac{\partial \zeta'}{\partial z}+2\frac{\partial^2 R^{-1}}{\partial z^2}\right)+\mu\nabla^2\zeta' = 0
\end{aligned}\right\} \quad \ldots\ldots(3).$$

These can be satisfied by assuming

$$\xi' = \alpha z \frac{\partial^2 R^{-1}}{\partial x \partial z}, \quad \eta' = \alpha z \frac{\partial^2 R^{-1}}{\partial y \partial z}, \quad \zeta' = \alpha z \frac{\partial^2 R^{-1}}{\partial z^2} \quad \ldots\ldots (4),$$

where α is a constant. For we find

$$\frac{\partial \xi'}{\partial x}+\frac{\partial \eta'}{\partial y}+\frac{\partial \zeta'}{\partial z} = \alpha\frac{\partial^2 R^{-1}}{\partial z^2},$$

$$\nabla^2\xi' = 2\alpha\frac{\partial}{\partial x}\frac{\partial^2 R^{-1}}{\partial z^2}, \quad \nabla^2\eta' = 2\alpha\frac{\partial}{\partial y}\frac{\partial^2 R^{-1}}{\partial z^2}, \quad \nabla^2\zeta' = 2\alpha\frac{\partial}{\partial z}\frac{\partial^2 R^{-1}}{\partial z^2};$$

and hence the three equations are of such forms as

$$[\alpha(\lambda+\mu)+2(\lambda+\mu)+2\mu\alpha]\frac{\partial}{\partial x}\frac{\partial^2 R^{-1}}{\partial z^2} = 0,$$

and they are all satisfied if $\alpha = -2(\lambda+\mu)/(\lambda+3\mu)$.

Hence we have

$$\left.\begin{aligned}
\xi_0 &= -\frac{\partial R^{-1}}{\partial x} - \frac{2(\lambda+\mu)}{\lambda+3\mu} z \frac{\partial^2 R^{-1}}{\partial x \partial z},\\
\eta_0 &= -\frac{\partial R^{-1}}{\partial y} - \frac{2(\lambda+\mu)}{\lambda+3\mu} z \frac{\partial^2 R^{-1}}{\partial y \partial z},\\
\zeta_0 &= \frac{\partial R^{-1}}{\partial z} - \frac{2(\lambda+\mu)}{\lambda+3\mu} z \frac{\partial^2 R^{-1}}{\partial z^2}
\end{aligned}\right\} \quad \ldots\ldots(5).$$

To find Δ from these, we have to calculate two sets of surface-tractions. Let Δ', ϖ_1', ϖ_2', ϖ_3' be the dilatation and rotations corresponding to any system of displacements u', v', w', and let F', G', H' be the corresponding surface-tractions. Then, if z be positive within the medium, the boundary-conditions are, by (15) of art. 29,

$$\left.\begin{aligned} F' &= -2\mu\frac{\partial u'}{\partial z} + 2\mu\varpi_2', \\ G' &= -2\mu\frac{\partial v'}{\partial z} - 2\mu\varpi_1', \\ H' &= -2\mu\frac{\partial w'}{\partial z} - \lambda\Delta' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The system F_0, G_0, H_0 is obtained by putting

$$u' = \frac{\partial r^{-1}}{\partial x}, \quad v' = \frac{\partial r^{-1}}{\partial y}, \quad w' = \frac{\partial r^{-1}}{\partial z},$$

and we get

$$\left.\begin{aligned} F_0 &= -2\mu\frac{\partial^2 r^{-1}}{\partial z \partial x}, \\ G_0 &= -2\mu\frac{\partial^2 r^{-1}}{\partial z \partial y}, \\ H_0 &= -2\mu\frac{\partial^2 r^{-1}}{\partial z^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The system L_0, M_0, N_0 is obtained by putting $u' = \xi_0$, $v' = \eta_0$, $w' = \zeta_0$. We get

$$\Delta' = \frac{4\mu}{\lambda + 3\mu}\frac{\partial^2 R^{-1}}{\partial z^2}, \quad \varpi_1' = \left(1 + \frac{\lambda + \mu}{\lambda + 3\mu}\right)\frac{\partial^2 R^{-1}}{\partial y \partial z},$$

$$\varpi_2' = -\left(1 + \frac{\lambda + \mu}{\lambda + 3\mu}\right)\frac{\partial^2 R^{-1}}{\partial z \partial x}.$$

Hence we find, when $z = 0$,

$$\left.\begin{aligned} L_0 &= 2\mu\frac{\lambda + \mu}{\lambda + 3\mu}\frac{\partial^2 R^{-1}}{\partial x \partial z} = -2\mu\frac{\lambda + \mu}{\lambda + 3\mu}\frac{\partial^2 r^{-1}}{\partial x \partial z}, \\ M_0 &= 2\mu\frac{\lambda + \mu}{\lambda + 3\mu}\frac{\partial^2 R^{-1}}{\partial y \partial z} = -2\mu\frac{\lambda + \mu}{\lambda + 3\mu}\frac{\partial^2 r^{-1}}{\partial y \partial z}, \\ N_0 &= -2\mu\frac{\lambda + \mu}{\lambda + 3\mu}\frac{\partial^2 R^{-1}}{\partial z^2} = -2\mu\frac{\lambda + \mu}{\lambda + 3\mu}\frac{\partial^2 r^{-1}}{\partial z^2} \end{aligned}\right\} \quad \ldots\ldots(8).$$

Hence, by (40) of art. 141, we find

$$\Delta = -\frac{\mu}{\pi(\lambda + 3\mu)}\iint\left(u\frac{\partial^2 r^{-1}}{\partial x \partial z} + v\frac{\partial^2 r^{-1}}{\partial y \partial z} + w\frac{\partial^2 r^{-1}}{\partial z^2}\right) dx dy \ldots\ldots(9).$$

This gives the value of Δ at (x', y', z') in terms of the given surface-displacements u, v, w.

145. Determination of the displacements.

We may now determine u, v, w at (x', y', z').

Let L, M, N denote the functions

$$L = \iint \frac{u}{r}\, dxdy, \quad M = \iint \frac{v}{r}\, dxdy, \quad N = \iint \frac{w}{r}\, dxdy \ldots\ldots (10),$$

which are finite, continuous and one-valued within the solid; then the value of Δ at (x', y', z') is given by the equation

$$\begin{aligned} -\frac{\lambda + 3\mu}{\mu}\, \pi\Delta &= \frac{\partial^2 L}{\partial x' \partial z'} + \frac{\partial^2 M}{\partial y' \partial z'} + \frac{\partial^2 N}{\partial z'^2} \\ &= \frac{\partial}{\partial z'}\left(\frac{\partial L}{\partial x'} + \frac{\partial M}{\partial y'} + \frac{\partial N}{\partial z'}\right) \\ &= \frac{\partial \phi}{\partial z'} \text{ say} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11), \end{aligned}$$

and the equations for u, v, w are three such as

$$\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} + \frac{\partial^2 u}{\partial z'^2} = \frac{\lambda + \mu}{(\lambda + 3\mu)\, \pi} \frac{\partial^2 \phi}{\partial x' \partial z'} \ldots\ldots\ldots\ldots (12).$$

Now L, M, N are the potentials of distributions of densities u, v, w on the surface, and therefore L, M, N, ϕ all satisfy Laplace's equation. Also the surface-value of u is $-\frac{1}{2\pi}\frac{\partial L}{\partial z'}$, for this is the density of the distribution whose potential is L. Thus we may take

$$\left.\begin{aligned} u &= -\frac{1}{2\pi}\frac{\partial L}{\partial z'} + \frac{1}{2\pi}\frac{\lambda + \mu}{\lambda + 3\mu} z' \frac{\partial \phi}{\partial x'}, \\ v &= -\frac{1}{2\pi}\frac{\partial M}{\partial z'} + \frac{1}{2\pi}\frac{\lambda + \mu}{\lambda + 3\mu} z' \frac{\partial \phi}{\partial y'}, \\ w &= -\frac{1}{2\pi}\frac{\partial N}{\partial z'} + \frac{1}{2\pi}\frac{\lambda + \mu}{\lambda + 3\mu} z' \frac{\partial \phi}{\partial z'} \end{aligned}\right\} \ldots\ldots\ldots (13),$$

where

$$\phi = \frac{\partial L}{\partial x'} + \frac{\partial M}{\partial y'} + \frac{\partial N}{\partial z'} \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

We shall devote the next seven articles to the discussion and generalisation of a particular example, returning in art. 153 to the problem of determining the displacements when the surface-tractions are given.

146. Particular Example.

The simplest example of these formulæ will be found by supposing that $L = M = 0$. Then $\dfrac{\partial N}{\partial z'} = \phi$, and ϕ satisfies

$$\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} = 0.$$

To fix ideas suppose the bounding plane horizontal, and the axis z drawn vertically downwards from a point in the plane. Then this example will correspond to the case when part of the bounding plane is vertically depressed, and the remainder held fixed.

Now ϕ is the potential of a distribution of matter on the surface, and the simplest example we can take is that of a single mass dm distributed over a small area $d\omega$ at the origin. (It is convenient to take this $-dm$.) We shall shew hereafter that dm is a constant multiple of the force required to depress the part of the surface near the origin.

Suppose then that

$$\phi = -\frac{dm}{r} \quad \text{.....(15),}$$

where r is the distance from the origin to any point of the solid. Since the only (x, y, z) that occurs is the origin, we may suppress the accents on (x', y', z') and write

$$\left.\begin{aligned} u &= \frac{dm}{2\pi} \frac{\lambda + \mu}{\lambda + 3\mu} \frac{zx}{r^3}, \\ v &= \frac{dm}{2\pi} \frac{\lambda + \mu}{\lambda + 3\mu} \frac{zy}{r^3}, \\ w &= \frac{dm}{2\pi r} + \frac{dm}{2\pi} \frac{\lambda + \mu}{\lambda + 3\mu} \frac{z^2}{r^3} \end{aligned}\right\} \quad \text{.....(16).}$$

If dm be regarded as a small finite quantity the depression near the origin is very great, and we must regard the origin as excluded from the part of the solid whose deformation we investigate. The problem is that of a considerable depression near a single point, and the above formulæ shew how to find the displacements at a distance from the point.

147. Elementary Discussion of Particular Example. Simple Solutions of First Type.

On account of its importance we shall consider this solution *à priori*.

It can be readily verified that the displacements

$$u = \frac{zx}{r^3}, \quad v = \frac{zy}{r^3}, \quad w = \frac{z^2}{r^3} + \frac{\lambda + 3\mu}{\lambda + \mu}\frac{1}{r} \quad \ldots\ldots\ldots(17),$$

where r is the distance of the point (x, y, z) from the origin, satisfy the general equations of equilibrium, when there is no bodily force, at all points not indefinitely near the origin. This is M. Boussinesq's *first type of simple solutions* of these equations.

Now these expressions can be written

$$u = -\frac{\partial^2 r}{\partial z \partial x}, \quad v = -\frac{\partial^2 r}{\partial z \partial y}, \quad w = -\frac{\partial^2 r}{\partial z^2} + \frac{\lambda + 2\mu}{\lambda + \mu}\nabla^2 r \ldots(18),$$

where r is the distance of (x, y, z) from a given point. If the above expressions be multiplied by any quantity independent of x, y, z we still have a solution, and the sum of any number of such solutions is a solution, and therefore

$$\left.\begin{aligned} u &= -\frac{\partial^2}{\partial z \partial x}\iint \rho_1 r dx' dy', \quad v = -\frac{\partial^2}{\partial z \partial y}\iint \rho_1 r dx' dy', \\ w &= -\frac{\partial^2}{\partial z^2}\iint \rho_1 r dx' dy' + \frac{\lambda + 2\mu}{\lambda + \mu}\nabla^2 \iint \rho_1 r dx' dy' \end{aligned}\right\} \ldots(19)$$

is a solution, r being the distance of (x, y, z) from the point $(x'y')$ on the plane $z = 0$, and ρ_1 any function of x', y'. Now we may regard ρ_1 as the surface-density of a distribution of matter on the plane $z = 0$, and then $\iint \rho_1 r dx' dy'$ is the "direct potential" of this distribution at (x, y, z), and, since $\nabla^2 r = 2/r$, $\frac{1}{2}\nabla^2 \iint \rho_1 r dx' dy'$ is the "inverse potential" (*i.e.* the ordinary gravitation potential) of this distribution.

148. Solid bounded by Infinite Plane. Purely Normal Surface Displacement.

We shall suppose the solid bounded by the plane $z = 0$, and seek the distribution of surface-traction which would produce the above system of displacements. It corresponds to purely normal displacement of a part of the bounding surface, the remainder being kept fixed.

It is easy to verify that the stresses T, S, R across any surface $z = \text{const.}$ arising from the displacements (17) are

$$\left.\begin{aligned} T &= -\frac{2\mu^2}{\lambda+\mu}\frac{x}{r^3} - 6\mu\frac{z^2x}{r^5}, \\ S &= -\frac{2\mu^2}{\lambda+\mu}\frac{y}{r^3} - 6\mu\frac{z^2y}{r^5}, \\ R &= -\frac{2\mu^2}{\lambda+\mu}\frac{z}{r^3} - 6\mu\frac{z^3}{r^5} \end{aligned}\right\} \quad \text{.....(20).}$$

The surface-tractions at $z = 0$, arising from the system (19) have a component H parallel to the z axis given by

$$H = \frac{2\mu^2}{\lambda+\mu}\iint\frac{z\rho_1 dx'\,dy'}{r^3} + 6\mu\iint\frac{z^3\rho_1 dx'\,dy'}{r^5} \quad \text{.....(21),}$$

the axis of z being drawn into the solid.

These quantities have finite limits when $z = 0$. The integral

$$-\iint\frac{z\rho_1 dx'\,dy'}{r^3}$$

is the attraction parallel to z of the surface distribution ρ_1, and therefore when $z = 0$ its limit is $-2\pi\rho_1$.

To find $\iint\frac{z^3\rho_1 dx'\,dy'}{r^5}$, we transform to polar coordinates r', θ' in the plane x', y', and put $r' = qz$, where q may be any positive quantity, thus this integral is

$$\int_0^{2\pi}\int_0^{\infty}\frac{\rho_1 q\,dq\,d\theta'}{(1+q^2)^{\frac{5}{2}}} = \tfrac{2}{3}\pi\rho_1.$$

Hence
$$H = \frac{\lambda+2\mu}{\lambda+\mu}4\mu\pi\rho_1 \quad \text{.....(22).}$$

The displacement at the surface is easily seen to be purely normal and equal to

$$\frac{\lambda+3\mu}{\lambda+\mu}\iint\frac{\rho_1 dx'\,dy'}{r} \quad \text{.....(23).}$$

Now suppose ρ_1 to vanish at all points except near the origin, and suppose that near the origin ρ_1 becomes infinite in such a way that $\iint\rho_1 dx'\,dy'$ is finite and equal to $\frac{\lambda+\mu}{\lambda+2\mu}\frac{W}{4\pi\mu}$.

Then the part near the origin suffers a very great normal displacement, and the resultant normal traction is W. If to fix ideas we consider the plane $z = 0$ horizontal, and the axis z drawn

vertically downwards into the solid, the problem is that of finding the deformations produced in the interior by very great normal pressure distributed over a very small area so as to have a finite resultant, and such tangential traction as will hold fixed the parts of the bounding plane at a distance from the origin.

To obtain the displacements in this problem we have to multiply the expressions (17) by $\frac{\lambda+\mu}{\lambda+2\mu}\frac{W}{4\pi\mu}$.

149. Weight supported at single point. Rest of surface fixed.

The displacement can be analysed into:

(1) a vertical displacement equal to $\frac{\lambda+3\mu}{\lambda+2\mu}\frac{W}{4\pi\mu}\frac{1}{r}$,

(2) a radial displacement from the origin equal to

$$\frac{\lambda+\mu}{\lambda+2\mu}\frac{W}{4\pi\mu}\frac{\cos\theta}{r^2},$$

where θ is the angle between the radius-vector and the vertical.

The stress exerted across any horizontal plane by the matter above it can be reduced to:

(1) a vertical pressure equal to

$$\frac{\mu W}{2\pi(\lambda+2\mu)}\frac{\cos\theta}{r^2}\left(1+3\frac{\lambda+\mu}{\mu}\cos^2\theta\right),$$

(2) a radial tangential traction outwards from the axis z equa

to $$\frac{\mu W}{2\pi(\lambda+2\mu)}\frac{\sin\theta}{r^2}\left(1+3\frac{\lambda+\mu}{\mu}\cos^2\theta\right).$$

At the surface these reduce to a radial tangential traction

$$\frac{\mu}{\lambda+2\mu}\frac{W}{2\pi r^2}$$

at all points at a finite distance from the origin. This is the traction required to hold the surface fixed.

To find the strains we refer to polar coordinates (θ, ϕ, r). The displacements u', v', w' along the meridian, the parallel, and the radius-vector are

$$u' = -\frac{\lambda+3\mu}{\lambda+2\mu}\frac{W}{4\pi\mu}\frac{\sin\theta}{r}, \quad v' = 0, \quad w' = \frac{W}{2\pi\mu}\frac{\cos\theta}{r} \ldots(24).$$

Then, using the formulæ of ch. VII. art. 125, we find that the

extension along the meridian is equal to that along the parallel, and either of them is

$$\tfrac{1}{4}(\lambda+\mu)\, Wr^{-2}\cos\theta/\{\pi\mu(\lambda+2\mu)\} \quad \ldots\ldots\ldots\ldots(25).$$

The *contraction* along the radius vector is

$$\tfrac{1}{2} Wr^{-2}\cos\theta/(\mu\pi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(26).$$

The cubical *compression* is

$$\tfrac{1}{2} Wr^{-2}\cos\theta/\{\pi(\lambda+2\mu)\} \ldots\ldots\ldots\ldots\ldots\ldots(27).$$

There is a shear in the meridian plane of amount

$$\tfrac{1}{2} Wr^{-2}\sin\theta/\{\pi(\lambda+2\mu)\} \ldots\ldots\ldots\ldots\ldots\ldots(28).$$

The axes of the elongation-quadric are in and perpendicular to the meridian plane, and the two in the meridian plane can be obtained by turning the tangent to the meridian (1) and the radius vector (3) through an angle $\frac{1}{2}\tan^{-1}\{2\mu\tan\theta/(3\lambda+5\mu)\}$ in the direction from (1) towards (3).

150. Generalisation. Particular Integral for the Bodily Forces[1].

The results of the preceding example are very important. We see that if the mass $\iint \rho_1 dx'\, dy'$ be very small and be distributed with a finite surface-density over a very small area, there will be a finite normal surface-traction per unit area near the origin[2], equal to

$$\frac{4\pi\mu(\lambda+2\mu)}{\lambda+\mu}(\text{surface-density}),$$

and vanishing surface-traction elsewhere. The displacements corresponding to this state of things are proportional to

$$-\frac{\partial^2 r}{\partial x\partial z}, \quad -\frac{\partial^2 r}{\partial y\partial z}, \quad -\frac{\partial^2 r}{\partial z^2}+\frac{\lambda+2\mu}{\lambda+\mu}\nabla^2 r.$$

We also found that if ρ_1 be the density of a surface-distribution on the plane $z=0$, the functions u, v, w given by

$$u=-\frac{\partial^2}{\partial x\partial z}\iint \rho_1 r\, dx'\, dy', \quad v=-\frac{\partial^2}{\partial y\partial z}\iint \rho_1 r\, dx'\, dy',$$

$$w=-\frac{\partial^2}{\partial z^2}\iint \rho_1 r\, dx'\, dy'+\frac{\lambda+2\mu}{\lambda+\mu}\nabla^2\iint \rho_1 r\, dx'\, dy',$$

[1] The methods of this and the following article are taken from M. Boussinesq's *Application des Potentiels &c.* pp. 276 sq.

[2] For the case of infinite normal surface-traction near the origin, having a finite resultant for a very small area, and vanishing surface-traction elsewhere see below, art. 162.

where r is the distance of any point (x, y, z) from the point (x', y') on the surface, are functions which satisfy the equations of equilibrium at every point on either side of the surface, ρ_1 being finite.

It follows from this that, if ρ' be the volume-density at (x', y', z') of a distribution of fictitious matter, and Φ be the "direct potential" of this distribution given by

$$\Phi = \iiint \rho' r dx' dy' dz',$$

the functions u, v, w given by

$$u = -\frac{\partial^2 \Phi}{\partial x \partial z}, \quad v = -\frac{\partial^2 \Phi}{\partial y \partial z}, \quad w = -\frac{\partial^2 \Phi}{\partial z^2} + \frac{\lambda + 2\mu}{\lambda + \mu} \nabla^2 \Phi \quad \ldots(29)$$

satisfy the differential equations of equilibrium, under no forces, at all points where ρ' vanishes.

To find the bodily forces X, Y, Z, which must be applied in order that the expressions given in (29) may continue to satisfy the equations of equilibrium at points where ρ' is finite, we form from the u, v, w of (29) the expressions such as

$$(\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u,$$

where $$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\mu}{\lambda + \mu} \frac{\partial}{\partial z} \nabla^2 \iiint \rho' r dx' dy' dz'.$$

Observing that $\nabla^4 \iiint \rho' r dx' dy' dz' = -8\pi\rho'$ when $r = 0$, we find $X' = 0$, $Y' = 0$, $\rho Z' = 8\pi\mu \dfrac{\lambda + 2\mu}{\lambda + \mu} \rho'$, where Z' is the value of Z at (x', y', z').

Thus the displacements

$$u = -\frac{\partial^2 \Phi}{\partial x \partial z}, \quad v = -\frac{\partial^2 \Phi}{\partial y \partial z}, \quad w = -\frac{\partial^2 \Phi}{\partial z^2} + \frac{\lambda + 2\mu}{\lambda + \mu} \nabla^2 \Phi$$

where $$\Phi = \frac{(\lambda + \mu) \rho}{8\pi\mu (\lambda + 2\mu)} \iiint r Z' dx' dy' dz' \quad \ldots\ldots\ldots\ldots(30),$$

correspond to a bodily force always parallel to the axis z.

Hence we can find the displacements, produced by any bodily forces whatever, in the form

$$\left.\begin{aligned} u &= \frac{\rho}{4\pi\mu}\iiint\left[\frac{X'}{r} - \frac{\lambda+\mu}{2(\lambda+2\mu)}\frac{\partial}{\partial x}\left(X'\frac{\partial r}{\partial x} + Y'\frac{\partial r}{\partial y} + Z'\frac{\partial r}{\partial z}\right)\right]dx'dy'dz', \\ v &= \frac{\rho}{4\pi\mu}\iiint\left[\frac{Y'}{r} - \frac{\lambda+\mu}{2(\lambda+2\mu)}\frac{\partial}{\partial y}\left(X'\frac{\partial r}{\partial x} + Y'\frac{\partial r}{\partial y} + Z'\frac{\partial r}{\partial z}\right)\right]dx'dy'dz', \\ w &= \frac{\rho}{4\pi\mu}\iiint\left[\frac{Z'}{r} - \frac{\lambda+\mu}{2(\lambda+2\mu)}\frac{\partial}{\partial z}\left(X'\frac{\partial r}{\partial x} + Y'\frac{\partial r}{\partial y} + Z'\frac{\partial r}{\partial z}\right)\right]dx'dy'dz' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(31).$$

These are the complete values of u, v, w at any point of an infinite solid to a finite part of which finite bodily forces X', Y', Z' are applied. They will also represent the displacements in such a solid, when the forces are applied at all points, provided they become at an infinite distance small of the order $\frac{1}{R}$ at least, where R is the distance of the infinitely distant (x, y, z) from the origin; and this condition will be satisfied if the bodily forces X, Y, Z at (x, y, z) are such that when R is infinite XR, YR, ZR converge uniformly to zero.

Another application of the results (31) is that they give particular integrals of the general equations of equilibrium of a finite solid mass subject to given bodily forces, whatever the surface-conditions may be. (See art. 137.)

151. Case of Force applied at single point.

Consider particularly the case of a single force parallel to the axis z applied at the origin. This force must be regarded as a bodily force $Z\rho dV$ acting on the element of mass ρdV. If we suppose Z to become infinite, while $Z\rho dV$ remains finite and $= P$ say, we have the limiting case of a force P applied at a single point (the origin). The displacements at any point not indefinitely near the origin are

$$\left.\begin{aligned} u &= \frac{\lambda+\mu}{\lambda+2\mu}\frac{P}{8\pi\mu}\frac{zx}{r^3}, \\ v &= \frac{\lambda+\mu}{\lambda+2\mu}\frac{P}{8\pi\mu}\frac{zy}{r^3}, \\ w &= \frac{\lambda+\mu}{\lambda+2\mu}\frac{P}{8\pi\mu}\frac{z^2}{r^3} + \frac{\lambda+3\mu}{\lambda+2\mu}\frac{P}{8\pi\mu}\frac{1}{r} \end{aligned}\right\} \quad \ldots\ldots\ldots(32).$$

If in art. 149 we write $\frac{1}{2}P$ for W, and take the vertical to mean the direction of the force P, all the statements of that article apply to this case.

152. Local Perturbations.

It is of great interest to enquire what will be the resulting displacements when a system of forces, which acting on a rigid body would produce equilibrium, is applied to a small part of a solid. In the Theory of Beams we have seen that Saint-Venant introduced a principle, which we have called the "Principle of the equivalence of statically equipollent loads". This principle states that the application of an equilibrating system of forces to a small part of the surface of a solid produces no sensible strain, except at very small distances from the part subjected to the action of the forces. M. Boussinesq brings this principle under a more general one which he states thus:

"External forces which produce equilibrium being applied to "an elastic solid at points within a given sphere provoke no "sensible displacement at distances from the sphere which are of "a certain order of magnitude in comparison with the radius."

M. Boussinesq has given several examples of this principle, and they lead to the conclusion that the application of forces to a small part of a solid produces, at sensible distances from the part, sensibly the same displacements as would be produced by the action of any other system of forces equivalent to the same resultant force and the same couple when applied to a rigid body. Near the region of application of the forces their mode of distribution sensibly affects the result, and the displacements differ finitely from those that would be produced by an equivalent set of forces differently distributed; but these deviations from the kind of displacement that depends on resultant forces and moments are practically confined to a small space near the region of application of the forces, and they are called by M. Boussinesq "Local Perturbations".

The student will find no difficulty in proving, by differentiating the formulæ of the last article with respect to z, that equal and opposite forces, applied at points near together, in the same straight line, produce at sensible distances displacements which vary directly as the forces, and as the distance between their points of application, and inversely as the square of the distance from the point of application of one of them; and that the resulting strains are directly as the forces and the distance between their points of application, and inversely as the cube of the dis-

tance from the point of application of one of them. Similar results can also be proved in the case where the forces are applied at points near together, but not in the same straight line. Such systems produce then displacements which can be regarded simply as local perturbations, insensible at sensible distances from the region within which they are applied. In the case of a long thin wire or rod strained by the application of forces at its ends, or a very thin plate or shell strained by forces applied at its edge, the falling off of the local perturbations at a little distance from the region of application of the force is likely to be much more rapid. The particular case of a very thin plate subjected to torsional couple has been considered in Thomson and Tait's *Natural Philosophy*, Part II., art. 728, where it is shewn that the local perturbations diminish according to an exponential function of the distance from the edge.

Displacement in a solid bounded by an infinite plane—Surface-tractions given.

153. Calculation of the Dilatation.

For the calculation of the cubical dilatation we must, according to art. 141, determine a system of displacements, which satisfy the equations of equilibrium and make the surface-tractions equal to

$$2\mu\frac{\partial^2 r^{-1}}{\partial z\partial x}, \quad 2\mu\frac{\partial^2 r^{-1}}{\partial z\partial y}, \quad 2\mu\frac{\partial^2 r^{-1}}{\partial z^2}, \text{ when } z=0.$$

This is the same system of surface-tractions as that which in the previous problem (art. 144) we denoted by L_0, M_0, N_0, except for a factor, viz.: these are $-(\lambda+3\mu)/(\lambda+\mu)$ times L_0, M_0, N_0, and thus the displacements which correspond to them are $-(\lambda+3\mu)/(\lambda+\mu)$ times the displacements ξ_0, η_0, ζ_0 of our previous problem (art. 144).

The displacements required have therefore the forms

$$\left.\begin{aligned} &\frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial R^{-1}}{\partial x}+2z\frac{\partial^2 R^{-1}}{\partial z\partial x},\\ &\frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial R^{-1}}{\partial y}+2z\frac{\partial^2 R^{-1}}{\partial z\partial y},\\ -&\frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial R^{-1}}{\partial z}+2z\frac{\partial^2 R^{-1}}{\partial z^2}, \end{aligned}\right\}$$

and the surface-values of these, when $z = 0$, are

$$\frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial r^{-1}}{\partial x}, \quad \frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial r^{-1}}{\partial y}, \quad \frac{\lambda+3\mu}{\lambda+\mu}\frac{\partial r^{-1}}{\partial z};$$

so that the value of Δ at (x', y', z') is given by the equation

$$\Delta = -\frac{1}{2\pi(\lambda+\mu)}\iint\left(F\frac{\partial r^{-1}}{\partial x} + G\frac{\partial r^{-1}}{\partial y} + H\frac{\partial r^{-1}}{\partial z}\right)dxdy \ldots (33),$$

where F, G, H are the given surface-tractions at $z = 0$.

154. Properties of certain functions.

The determination of the rotations is more difficult and depends upon the properties of the function χ defined by the equation

$$\chi = \log(z + z' + R) \ldots\ldots\ldots\ldots\ldots\ldots (34).$$

This function is finite, continuous, and one-valued within the solid, and satisfies Laplace's equation.

We have

$$\frac{\partial\chi}{\partial x} = \frac{1}{z+z'+R}\frac{x-x_1}{R},$$

$$\frac{\partial\chi}{\partial y} = \frac{1}{z+z'+R}\frac{y-y_1}{R},$$

$$\frac{\partial\chi}{\partial z} = \frac{1}{z+z'+R}\frac{z-z_1}{R} + \frac{1}{z+z'+R} = \frac{1}{R},$$

$$\frac{\partial^2\chi}{\partial x^2} = \frac{R^2-(x-x_1)^2}{R^3(z+z'+R)} - \frac{(x-x_1)^2}{R^2(z+z'+R)^2}$$

$$= \frac{1}{R(z+z'+R)} - \frac{(z+z'+2R)(x-x_1)^2}{R^3(z+z'+R)^2},$$

$$\frac{\partial^2\chi}{\partial y^2} = \frac{1}{R(z+z'+R)} - \frac{(z+z'+2R)(y-y_1)^2}{R^3(z+z'+R)^2},$$

$$\frac{\partial^2\chi}{\partial z^2} = -\frac{z-z_1}{R^3} = \frac{1}{R(z+z'+R)} - \frac{(z+z'+2R)(z-z_1)^2}{R^3(z+z'+R)^2}$$

$$- \frac{1}{(z+z'+R)^2}\left\{1 + 2\frac{z-z_1}{R}\right\}.$$

Hence

$$\nabla^2\chi = \frac{3}{R(z+z'+R)} - \frac{(z+z'+2R)}{R(z+z'+R)^2} - \frac{2(z+z')+R}{R(z+z'+R)^2} = 0.$$

Also
$$\frac{\partial^2\chi}{\partial x\partial z}=\frac{\partial R^{-1}}{\partial x},$$
$$\frac{\partial^2\chi}{\partial y\partial z}=\frac{\partial R^{-1}}{\partial y},$$
$$\frac{\partial^2\chi}{\partial z^2}=\frac{\partial R^{-1}}{\partial z}.$$

Again, consider the function ψ defined by the equation
$$\psi=(z+z')\log(z+z'+R)-R \quad \text{.....(35).}$$

We find
$$\frac{\partial\psi}{\partial z'}=\frac{\partial\psi}{\partial z}=\chi,$$

and ψ, $\frac{\partial\psi}{\partial z'}$ and $\frac{\partial\psi}{\partial z}$ satisfy Laplace's equation.

155. Determination of subsidiary displacements required in finding the rotations.

To find ϖ_1 we have, by art. 142, to find displacements ξ_1, η_1, ζ_1 satisfying the equations of equilibrium, and such that the surface-tractions that would produce them are the same as if the displacements near the surface were $0, -\frac{\partial r^{-1}}{\partial z}, \frac{\partial r^{-1}}{\partial y}$; thus we have to find displacements ξ_1, η_1, ζ_1 which satisfy the boundary-conditions

$$\left.\begin{aligned}
&\mu\left(\frac{\partial\xi_1}{\partial z}+\frac{\partial\zeta_1}{\partial x}\right)=\mu\frac{\partial^2 r^{-1}}{\partial y\partial z},\\
&\mu\left(\frac{\partial\zeta_1}{\partial y}+\frac{\partial\eta_1}{\partial z}\right)=\mu\left(\frac{\partial^2 r^{-1}}{\partial y^2}-\frac{\partial^2 r^{-1}}{\partial z^2}\right),\\
&\lambda\left(\frac{\partial\xi_1}{\partial x}+\frac{\partial\eta_1}{\partial y}+\frac{\partial\zeta_1}{\partial z}\right)+2\mu\frac{\partial\zeta_1}{\partial z}=2\mu\frac{\partial^2 r^{-1}}{\partial y\partial z}
\end{aligned}\right\} \quad \text{.....(36).}$$

The functions $0, -\frac{\partial R^{-1}}{\partial z}, \frac{\partial R^{-1}}{\partial y}$ satisfy the first two conditions identically, and therefore we take
$$\xi_1=\xi_1', \quad \eta_1=\eta_1'-\frac{\partial R^{-1}}{\partial z}, \quad \zeta_1=\zeta_1'+\frac{\partial R^{-1}}{\partial y},$$

and the functions ξ_1', η_1', ζ_1' must be finite, continuous, and one-valued throughout the solid, and must satisfy certain differential equations, to be given presently, and the boundary-conditions
$$\frac{\partial\xi_1'}{\partial z}+\frac{\partial\zeta_1'}{\partial x}=0, \quad \frac{\partial\zeta_1'}{\partial y}+\frac{\partial\eta_1'}{\partial z}=0 \quad \text{.....(37),}$$

and
$$\lambda\left(\frac{\partial\xi_1'}{\partial x}+\frac{\partial\eta_1'}{\partial y}+\frac{\partial\zeta_1'}{\partial z}\right)+2\mu\frac{\partial\zeta_1'}{\partial z}=-4\mu\frac{\partial^2 R^{-1}}{\partial y\partial z} \quad \text{.....(38).}$$

Now supposing that

$$\xi_1' = \frac{\partial \xi''}{\partial y}, \quad \eta_1' = \frac{\partial \eta''}{\partial y}, \quad \zeta_1' = \frac{\partial \zeta''}{\partial y} \qquad \text{.....(39),}$$

the third of these boundary-conditions becomes

$$\lambda \left(\frac{\partial \xi''}{\partial x} + \frac{\partial \eta''}{\partial y} \right) + (\lambda + 2\mu) \frac{\partial \zeta''}{\partial z} = -4\mu \frac{\partial R^{-1}}{\partial z} \qquad \text{.....(40).}$$

In the notation of the last article these equations can be satisfied by assuming

$$\left.\begin{aligned} \xi'' &= 2z \frac{\partial^2 \chi}{\partial x \partial z} + \alpha \frac{\partial \chi}{\partial x}, \\ \eta'' &= 2z \frac{\partial^2 \chi}{\partial y \partial z} + \alpha \frac{\partial \chi}{\partial y}, \\ \zeta'' &= 2z \frac{\partial^2 \chi}{\partial z^2} + \beta \frac{\partial \chi}{\partial z} \end{aligned}\right\} \qquad \text{.....(41),}$$

where α and β are constants: for with these values we find

$$\frac{\partial \xi''}{\partial x} + \frac{\partial \eta''}{\partial y} + \frac{\partial \zeta''}{\partial z} = 2 \frac{\partial^2 \chi}{\partial z^2} - (\alpha - \beta) \frac{\partial^2 \chi}{\partial z^2},$$

$$\frac{\partial \zeta''}{\partial z} = (2 + \beta) \frac{\partial^2 \chi}{\partial z^2} + 2z \frac{\partial^3 \chi}{\partial z^3};$$

and, when $z = 0$, we have

$$\lambda \left(\frac{\partial \xi''}{\partial x} + \frac{\partial \eta''}{\partial y} + \frac{\partial \zeta''}{\partial z} \right) + 2\mu \frac{\partial \zeta''}{\partial z} = [(2 - \alpha + \beta)\lambda + 2(2 + \beta)\mu] \frac{\partial^2 \chi}{\partial z^2};$$

and since $\dfrac{\partial^2 \chi}{\partial z^2} = \dfrac{\partial R^{-1}}{\partial z}$, the third of the boundary-conditions (40) is satisfied if

$$-\alpha\lambda + (\beta + 2)(\lambda + 2\mu) = -4\mu.$$

The other two boundary-conditions become, when $z = 0$,

$$(\alpha + \beta + 2) \frac{\partial^2 \chi}{\partial x \partial z} = 0, \quad (\alpha + \beta + 2) \frac{\partial^2 \chi}{\partial y \partial z} = 0.$$

Hence $$\alpha + \beta + 2 = 0.$$

Thus $$\alpha = \frac{2\mu}{\lambda + \mu}, \quad \beta = -2 \frac{\lambda + 2\mu}{\lambda + \mu}.$$

It follows that the displacements

$$\left.\begin{aligned}
\xi_1 &= 2z\frac{\partial^3\chi}{\partial x\partial y\partial z} + 2\frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial x\partial y},\\
\eta_1 &= 2z\frac{\partial^3\chi}{\partial y^2\partial z} + 2\frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial y^2} - \frac{\partial R^{-1}}{\partial z}\\
&= 2z\frac{\partial^3\chi}{\partial y^2\partial z} + 2\frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial y^2} - \frac{\partial^2\chi}{\partial z^2},\\
\zeta_1 &= 2z\frac{\partial^3\chi}{\partial y\partial z^2} - 2\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial y\partial z} + \frac{\partial R^{-1}}{\partial y}\\
&= 2z\frac{\partial^3\chi}{\partial y\partial z^2} - 2\frac{\mu}{\lambda+\mu}\frac{\partial^2\chi}{\partial y\partial z} - \frac{\partial^2\chi}{\partial y\partial z}
\end{aligned}\right\} \quad \text{.....(42)}$$

satisfy the boundary-conditions. It is easy to verify that they also satisfy the differential equations of equilibrium.

156. Calculation of the Rotations.

Hence we find, by art. 142, and remembering that $\nabla^2\chi = 0$,

$$4\pi\mu\varpi_1 = \frac{\lambda+2\mu}{\lambda+\mu}\iint\left(F\frac{\partial^2\chi}{\partial x\partial y} + G\frac{\partial^2\chi}{\partial y^2} - H\frac{\partial^2\chi}{\partial y\partial z}\right)dxdy + \iint\left(G\frac{\partial^2\chi}{\partial x^2} - F\frac{\partial^2\chi}{\partial x\partial y}\right)dxdy \quad \text{.....(43).}$$

In like manner we should find

$$4\pi\mu\varpi_2 = -\frac{\lambda+2\mu}{\lambda+\mu}\iint\left(F\frac{\partial^2\chi}{\partial x^2} + G\frac{\partial^2\chi}{\partial x\partial y} - H\frac{\partial^2\chi}{\partial x\partial z}\right)dxdy + \iint\left(G\frac{\partial^2\chi}{\partial x\partial y} - F\frac{\partial^2\chi}{\partial y^2}\right)dxdy \quad \text{.....(44).}$$

It is easy to shew that the functions ξ_3, η_3, ζ_3 required for the determination of ϖ_3 are

$$\xi_3 = \frac{\partial R^{-1}}{\partial y}, \quad \eta_3 = -\frac{\partial R^{-1}}{\partial x}, \quad \zeta_3 = 0;$$

and therefore

$$4\pi\mu\varpi_3 = \iint\left(F\frac{\partial^2\chi}{\partial y\partial z} - G\frac{\partial^2\chi}{\partial x\partial z}\right)dxdy \quad \text{.....(45).}$$

157. Simplified forms for the Dilatation and the Rotations.

We introduce now four functions L, M, N, ϕ defined as follows:

$$\left.\begin{aligned} L &= \iint F\chi\,dx\,dy, \\ M &= \iint G\chi\,dx\,dy, \\ N &= \iint H\chi\,dx\,dy, \\ \phi &= \frac{\partial L}{\partial x'} + \frac{\partial M}{\partial y'} + \frac{\partial N}{\partial z'} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (46).$$

Then since

$$\frac{\partial \chi}{\partial x'} = -\frac{\partial \chi}{\partial x}, \quad \frac{\partial \chi}{\partial y'} = -\frac{\partial \chi}{\partial y}, \quad \frac{\partial \chi}{\partial z'} = \frac{\partial \chi}{\partial z}$$

and

$$\nabla^2 \chi = 0,$$

it follows that L, M, N, ϕ all satisfy the equation

$$\frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2} + \frac{\partial^2 V}{\partial z'^2} = 0$$

at all points within the solid, and are finite, continuous, and one-valued functions of x', y', z'.

Now the value of Δ given in art. 153 and the values found in the last article for ϖ_1, ϖ_2, ϖ_3 can be re-written in the following forms:

$$\left.\begin{aligned} \Delta &= \frac{1}{2\pi(\lambda+\mu)} \frac{\partial \phi}{\partial z'}, \\ 2\varpi_1 &= \frac{\lambda+2\mu}{2\pi\mu(\lambda+\mu)} \frac{\partial \phi}{\partial y'} + \frac{1}{2\pi\mu} \frac{\partial}{\partial x'} \left(\frac{\partial M}{\partial x'} - \frac{\partial L}{\partial y'} \right), \\ 2\varpi_2 &= -\frac{\lambda+2\mu}{2\pi\mu(\lambda+\mu)} \frac{\partial \phi}{\partial x'} + \frac{1}{2\pi\mu} \frac{\partial}{\partial y'} \left(\frac{\partial M}{\partial x'} - \frac{\partial L}{\partial y'} \right), \\ 2\varpi_3 &= \frac{1}{2\pi\mu} \frac{\partial}{\partial z'} \left(\frac{\partial M}{\partial x'} - \frac{\partial L}{\partial y'} \right) \end{aligned}\right\} \quad \ldots\ldots (47).$$

158. Determination of the Displacement w.

To find u, v, w as functions of (x', y', z') we have to find solutions of such equations as

$$\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} + \frac{\partial^2 u}{\partial z'^2} = -\frac{\lambda+\mu}{\mu} \frac{\partial \Delta}{\partial x'} \quad \ldots\ldots\ldots\ldots (48),$$

with the boundary-conditions

$$\left.\begin{aligned} -2\mu \frac{\partial u}{\partial z'} + 2\mu\varpi_2 &= F, \\ -2\mu \frac{\partial v}{\partial z'} - 2\mu\varpi_1 &= G, \\ -2\mu \frac{\partial w}{\partial z'} - \lambda\Delta &= H \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (49)$$

when $z' = 0$.

The determination of w is comparatively simple. It has to satisfy the equation

$$\frac{\partial^2 w}{\partial x'^2}+\frac{\partial^2 w}{\partial y'^2}+\frac{\partial^2 w}{\partial z'^2}=-\frac{1}{2\pi\mu}\frac{\partial^2 \phi}{\partial z'^2} \quad \text{.....(50),}$$

and the boundary-condition

$$\frac{\partial w}{\partial z'}=-\frac{H}{2\mu}-\frac{\lambda}{4\pi\mu(\lambda+\mu)}\frac{\partial \phi}{\partial z'} \quad \text{.....(51)}$$

when $z'=0$.

A particular solution of the differential equation (50) is

$$w=-\frac{1}{4\pi\mu}z'\frac{\partial \phi}{\partial z'},$$

and this makes

$$\frac{\partial w}{\partial z'}=-\frac{1}{4\pi\mu}\frac{\partial \phi}{\partial z'}$$

when $z'=0$.

We have to add complementary solutions which make

$$\frac{\partial w}{\partial z'}=-\frac{H}{2\mu}, \text{ and } \frac{\partial w}{\partial z'}=\frac{1}{4\pi(\lambda+\mu)}\frac{\partial \phi}{\partial z'}$$

respectively, when $z'=0$, and these are

$$\frac{1}{4\pi\mu}\frac{\partial N}{\partial z'}, \text{ and } \frac{\phi}{4\pi(\lambda+\mu)}.$$

Hence the complete value of w is

$$w=\frac{1}{4\pi\mu}\frac{\partial N}{\partial z'}+\frac{\phi}{4\pi(\lambda+\mu)}-\frac{z'}{4\pi\mu}\frac{\partial \phi}{\partial z'} \quad \text{.....(52).}$$

159. Determination of the Displacements u and v.

The form for w suggests that for u and v we should take

$$\left.\begin{aligned} u&=u'+\frac{1}{4\pi\mu}\frac{\partial L}{\partial z'}-\frac{z'}{4\pi\mu}\frac{\partial \phi}{\partial x'},\\ v&=v'+\frac{1}{4\pi\mu}\frac{\partial M}{\partial z'}-\frac{z'}{4\pi\mu}\frac{\partial \phi}{\partial y'} \end{aligned}\right\} \quad \text{.....(53).}$$

Then u' and v' must satisfy the equation

$$\frac{\partial^2 V}{\partial x'^2}+\frac{\partial^2 V}{\partial y'^2}+\frac{\partial^2 V}{\partial z'^2}=0,$$

and the boundary-conditions

$$\left.\begin{aligned}\frac{\partial u'}{\partial z'} &= -\frac{1}{4\pi(\lambda+\mu)}\frac{\partial \phi}{\partial x'} + \frac{1}{4\pi\mu}\frac{\partial}{\partial y'}\left(\frac{\partial M}{\partial x'} - \frac{\partial L}{\partial y'}\right),\\ \frac{\partial v'}{\partial z'} &= -\frac{1}{4\pi(\lambda+\mu)}\frac{\partial \phi}{\partial y'} - \frac{1}{4\pi\mu}\frac{\partial}{\partial x'}\left(\frac{\partial M}{\partial x'} - \frac{\partial L}{\partial y'}\right)\end{aligned}\right\} \quad \ldots(54)$$

when $z' = 0$.

We introduce now four new functions L', M', N', ϕ' defined by the equations

$$\left.\begin{aligned} L' &= \iint F\psi\, dx\, dy,\\ M' &= \iint G\psi\, dx\, dy,\\ N' &= \iint H\psi\, dx\, dy,\\ \phi' &= \frac{\partial L'}{\partial x'} + \frac{\partial M'}{\partial y'} + \frac{\partial N'}{\partial z'}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(55),$$

where ψ is the function $(z+z')\log(z+z'+R) - R$ defined in art. 154 and possessing the property

$$\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial z'} = \chi.$$

We deduce

$$\frac{\partial L'}{\partial z'} = L, \quad \frac{\partial M'}{\partial z'} = M, \quad \frac{\partial N'}{\partial z'} = N, \quad \frac{\partial \phi'}{\partial z'} = \phi.$$

The boundary-conditions become

$$\left.\begin{aligned} u' &= -\frac{1}{4\pi(\lambda+\mu)}\frac{\partial \phi'}{\partial x'} + \frac{1}{4\pi\mu}\frac{\partial}{\partial y'}\left(\frac{\partial M'}{\partial x'} - \frac{\partial L'}{\partial y'}\right),\\ v' &= -\frac{1}{4\pi(\lambda+\mu)}\frac{\partial \phi'}{\partial y'} - \frac{1}{4\pi\mu}\frac{\partial}{\partial x'}\left(\frac{\partial M'}{\partial x'} - \frac{\partial L'}{\partial y'}\right)\end{aligned}\right\} \quad \ldots\ldots(56).$$

Since L', M', N', ϕ' are finite, continuous, and one-valued within the solid, and satisfy the equation

$$\frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2} + \frac{\partial^2 V}{\partial z'^2} = 0,$$

we conclude that these values of u', v' also satisfy this differential equation, and, since they satisfy the boundary-conditions, we conclude that the complete values of u and v are

$$\left.\begin{aligned} u &= \frac{1}{4\pi\mu}\frac{\partial L}{\partial z'} - \frac{z'}{4\pi\mu}\frac{\partial \phi}{\partial x'} - \frac{1}{4\pi(\lambda+\mu)}\frac{\partial \phi'}{\partial x'} + \frac{1}{4\pi\mu}\frac{\partial}{\partial y'}\left(\frac{\partial M'}{\partial x'} - \frac{\partial L'}{\partial y'}\right),\\ v &= \frac{1}{4\pi\mu}\frac{\partial M}{\partial z'} - \frac{z'}{4\pi\mu}\frac{\partial \phi}{\partial y'} - \frac{1}{4\pi(\lambda+\mu)}\frac{\partial \phi'}{\partial y'} - \frac{1}{4\pi\mu}\frac{\partial}{\partial x'}\left(\frac{\partial M'}{\partial x'} - \frac{\partial L'}{\partial y'}\right)\end{aligned}\right\} \quad \ldots(57).$$

Thus the displacements u, v, w are completely determined, in terms of the functions L, M, N, ϕ, L', M', N', ϕ' introduced and defined by equations (46) and (55).

160. Particular example.

The simplest example of these formulæ will be found by supposing the surface-traction to be purely normal. Then if, as in art. 149, we take the bounding plane horizontal and the axis z vertically downwards, this example corresponds to the case where the plane supports a weight distributed over its surface.

We shall proceed with the example of a single weight W, supposed distributed over a small area $d\omega$ at the origin, and we shall take

$$W = \rho_1 d\omega,$$

so that ρ_1 is the weight per unit area supported at the origin, and therefore $H = \rho_1$ near the origin, and $H = 0$ elsewhere.

Then the functions L, M, N, ϕ of the previous work are as follows

$$L = 0, \quad M = 0, \quad N = W \log (z + r), \quad \phi = \frac{W}{r} = \frac{\partial N}{\partial z}$$

and the functions L', M', N', ϕ' are

$$L' = 0, \quad M' = 0, \quad N' = W \{z \log (z + r) - r\}, \quad \phi' = W \log (z + r) = N$$

where, as in art. 146, we have changed the notation, since the only (x, y, z) that occurs is the origin, and have suppressed the accents on (x', y', z').

The displacements are

$$\left.\begin{aligned} u &= -\frac{W}{4\pi(\lambda+\mu)} \frac{x}{r(z+r)} + \frac{W}{4\pi\mu} \frac{zx}{r^3}, \\ v &= -\frac{W}{4\pi(\lambda+\mu)} \frac{y}{r(z+r)} + \frac{W}{4\pi\mu} \frac{zy}{r^3}, \\ w &= \frac{W(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)} \frac{1}{r} + \frac{W}{4\pi\mu} \frac{z^2}{r^3} \end{aligned}\right\} \quad \ldots\ldots\ldots (58).$$

We shall give an elementary discussion of the results.

161. Simple Solutions of Second Type.

It can be readily verified that the displacements

$$u = \frac{x}{r(z+r)}, \quad v = \frac{y}{r(z+r)}, \quad w = \frac{1}{r} \quad \ldots\ldots\ldots (59),$$

where r is the distance of the point (x, y, z) from the origin, satisfy the general equations of equilibrium at all points, not indefinitely near the origin, which lie on the side z positive of the plane $z = 0$. They constitute M. Boussinesq's *second type of simple solutions.*

Now these may be written

$$u = \frac{\partial}{\partial x} \log (z + r), \quad v = \frac{\partial}{\partial y} \log (z + r), \quad w = \frac{\partial}{\partial z} \log (z + r) \ldots (60),$$

and, generalising as in art. 147, we may conclude that, if X be the "logarithmic potential" of a distribution of surface-density ρ_1 on the plane $z = 0$, given by

$$X = \iint \rho_1 \log (z + r)\, dx'dy' \ldots\ldots\ldots\ldots\ldots (61),$$

where r is the distance of (x, y, z) from $(x', y', 0)$, then the displacements

$$u = \frac{\partial X}{\partial x}, \quad v = \frac{\partial X}{\partial y}, \quad w = \frac{\partial X}{\partial z} \ldots\ldots\ldots\ldots (62)$$

satisfy the equations of equilibrium.

We may verify at once by differentiation that $\nabla^2 \log (z + r) = 0$, and therefore that $\nabla^2 X = 0$; also that $\dfrac{\partial X}{\partial z}$ is the ordinary inverse potential $\iint \dfrac{\rho_1}{r}\, dx'dy'$ of the distribution ρ_1.

The system of displacements (60) is a system for which the dilatation is zero, and we easily find for the stresses T, S, R across any plane parallel to the bounding surface

$$T = -\frac{2\mu x}{r^3}, \quad S = -\frac{2\mu y}{r^3}, \quad R = -\frac{2\mu z}{r^3} \ldots\ldots (63).$$

162. Weight supported at single point. Rest of Surface Free.

Now, comparing these stresses (63) with the stresses found in (20) of art. 148, we see that we can reduce the tangential stress to zero at all points of the plane $z = 0$, by taking for the displacements certain multiples of those of the first type, compounded with certain multiples of those of the second type.

Take then for the displacements the product of $W/4\pi\mu$ and the simple solutions of the first type given by (17) of art. 147, viz.:

$$u = \frac{zx}{r^3}, \quad v = \frac{zy}{r^3}, \quad w = \frac{z^2}{r^3} + \frac{\lambda + 3\mu}{\lambda + \mu} \frac{1}{r},$$

and the product of $-W/4\pi(\lambda+\mu)$ and the simple solutions of the second type given by (59), thus we have

$$\left.\begin{aligned} u &= -\frac{W}{4\pi(\lambda+\mu)}\frac{x}{r(z+r)} + \frac{W}{4\pi\mu}\frac{zx}{r^3}, \\ v &= -\frac{W}{4\pi(\lambda+\mu)}\frac{y}{r(z+r)} + \frac{W}{4\pi\mu}\frac{zy}{r^3}, \\ w &= \frac{W(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\frac{1}{r} + \frac{W}{4\pi\mu}\frac{z^2}{r^3} \end{aligned}\right\} \ldots\ldots\ldots (64).$$

These correspond to displacements produced by a single weight W supported at the origin; for the surface-tractions on any plane $z = \text{const.}$ are

$$F = \frac{3W}{2\pi}\frac{z^2x}{r^5}, \quad G = \frac{3W}{2\pi}\frac{z^2y}{r^5}, \quad H = \frac{3W}{2\pi}\frac{z^3}{r^5} \ldots\ldots\ldots (65).$$

The resultant of H has a limiting value when $z = 0$, which is the limit of

$$\frac{3}{2\pi}\iint\frac{z^3}{r^5}\rho_1\,dx'dy' \ldots\ldots\ldots\ldots\ldots\ldots (66),$$

where $\iint\rho_1\,dx'dy' = W$. As in art. 148 we find $H = \rho_1$, and the resultant of H over the very small surface to which it is applied is W.

The stress at any point across any plane parallel to the surface is in the direction joining the point to the origin (where the weight is supported), and is, as it were, a repulsion from that point of amount

$$\frac{3W}{2\pi r^2}\cos^2\theta,$$

where θ is the angle the radius-vector makes with the vertical.

If we describe a sphere to pass through the origin and the point (x, y, z), and to have its centre on the axis of z, and if D be the diameter of this sphere, we shall have $r^2 = Dz$, and the stress across horizontal planes will be the same at all points of such a sphere, and its amount is

$$\frac{3W}{2\pi D^2}.$$

We notice that the expressions for the stresses (65) do not contain any elastic constant, so that the transmission of force across the horizontal planes is of the same character for all isotropic solids.

The horizontal displacement is along the radius perpendicular to the axis z, and is equal to

$$\frac{W \sin\theta}{4\pi\mu r}\left[\cos\theta - \frac{\mu}{\lambda+\mu}\frac{1}{1+\cos\theta}\right],$$

where θ is the angle the radius-vector from the origin makes with the axis z.

Within the cone whose generators are given by the equation

$$\cos^2\theta + \cos\theta = \mu/(\lambda+\mu),$$

this displacement is from the axis, and without this cone it is towards the axis. When $\lambda = \mu$ the angle of this separating cone is $\cos^{-1}\frac{1}{2}(\sqrt{3}-1)$ or $68^\circ\,32'$ nearly.

The vertical displacement is

$$\frac{W}{4\pi\mu r}\left(\frac{\lambda+2\mu}{\lambda+\mu} + \cos^2\theta\right),$$

and is always downwards.

At the surface the vertical displacement downwards is

$$\tfrac{1}{4}\, Wr^{-1}(\lambda+2\mu)/\{\mu\pi(\lambda+\mu)\},$$

and the horizontal displacement towards the origin is

$$\tfrac{1}{4}\, Wr^{-1}/\{\pi(\lambda+\mu)\}.$$

The form assumed by the free surface is approximately the surface formed by the revolution round the axis z of the hyperbola

$$4\pi\mu(\lambda+\mu)\, zx = (\lambda+2\mu)\, W.$$

163. Weight distributed in any manner on Surface.

In general taking, in the notation of art. 150, Φ for the direct potential of the distribution ρ_1, and X for the logarithmic potential of the same distribution on $z=0$, and compounding $\dfrac{1}{4\pi\mu}$ of the displacements of the first type given by (29), viz.:

$$u = -\frac{\partial^2\Phi}{\partial z\partial x}, \quad v = -\frac{\partial^2\Phi}{\partial y\partial z}, \quad w = -\frac{\partial^2\Phi}{\partial z^2} + \frac{\lambda+2\mu}{\lambda+\mu}\nabla^2\Phi,$$

and $-\dfrac{1}{4\pi(\lambda+\mu)}$ of the displacements of the second type given by (62), viz.:

$$u = \frac{\partial X}{\partial x}, \quad v = \frac{\partial X}{\partial y}, \quad w = \frac{\partial X}{\partial z},$$

we obtain displacements

$$\left.\begin{aligned} u &= -\frac{1}{4\pi(\lambda+\mu)}\frac{\partial X}{\partial x} - \frac{1}{4\pi\mu}\frac{\partial^2\Phi}{\partial z\partial x}, \\ v &= -\frac{1}{4\pi(\lambda+\mu)}\frac{\partial X}{\partial y} - \frac{1}{4\pi\mu}\frac{\partial^2\Phi}{\partial z\partial y}, \\ w &= -\frac{1}{4\pi(\lambda+\mu)}\frac{\partial X}{\partial z} - \frac{1}{4\pi\mu}\frac{\partial^2\Phi}{\partial z^2} + \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\nabla^2\Phi \end{aligned}\right\} \ldots(67),$$

where

$$\left.\begin{aligned} X &= \iint \rho_1 \log(z+r)\, dx'dy', \\ \Phi &= \iint \rho_1 r\, dx'dy' \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(68);$$

and these are the displacements produced by purely normal surface-traction ρ_1 per unit area applied at $z = 0$.

M. Boussinesq has given several examples of the application of these formulæ to determine the displacements produced in a solid bounded by a horizontal plane which supports a load distributed in a given manner.

CHAPTER X.

LAMÉ'S PROBLEM [1].

164. Statement of the Problem.

Lamé was the first to solve the problem of determining the displacements in an elastic sphere or spherical shell whose surface is subject to any system of tractions, and whose particles attract each other according to the Law of Gravitation. Sir W. Thomson has considered the more general problem where the sphere is subject also to the action of forces having a potential which satisfies Laplace's equation. The most general problem of the kind which has been solved is as follows:

A gravitating solid elastic sphere, of homogeneous isotropic material, is rotating slowly about a diameter, and is subject to the action of bodily forces derivable from a potential expressible in spherical harmonic series; it is required to determine the resulting displacements.

We shall begin with the problem of the elastic equilibrium of the sphere when there is no bodily force, and the displacement at any point of the surface is a given function of position on the surface. We shall then proceed to the same problem when the

[1] The following among other authorities may be consulted:

Lamé, *Léçons sur les Coordonnées Curvilignes.*

Thomson and Tait, *Natural Philosophy*, Part II.

Sir W. Thomson, *Mathematical and Physical Papers*, Vol. III.

G. H. Darwin, 'On the Stresses produced in the interior of the Earth by the Weight of Continents and Mountains'. *Phil. Trans. R. S.* 1882, and 'On the Dynamical Theory of the Tides of long period'. *Proc. R. S.* 1886.

Chree, 'On the Equations of an Isotropic Elastic Solid in Cylindrical and Polar Coordinates', and 'On the Stresses in rotating Spherical Shells'. *Camb. Phil. Soc. Trans.* XIV., 1889. 'A new solution of the equations of an isotropic elastic solid...'. *Quarterly Journal*, 1886 and 1888, and 'Some Applications of Physics and Mathematics to Geology', *Phil. Mag.* XXXII., 1891.

surface-tractions are given. Finally we shall investigate the general problem. The general solution of the problem of elastic equilibrium of a spherical shell with given displacements or surface-tractions at the inner and outer surfaces is very complicated, and the reader is referred for it to Thomson and Tait's *Natural Philosophy*. We shall consider only the particular case of a spherical cavity in an infinite solid, which has an important practical application.

165. The sphere with given surface-displacements.

We have to find solutions of the equations

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u=0,\\ (\lambda+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 v=0,\\ (\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

where

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which are finite, continuous, and one-valued within a sphere of radius a, and make u, v, w given functions of position on the surface.

We may suppose the given surface-values of u, v, w expanded in spherical surface-harmonics, and thus we may take at the surface

$$u=\sum_{n=1}^{n=\infty} A_n, \quad v=\sum_{n=1}^{n=\infty} B_n, \quad w=\sum_{n=1}^{n=\infty} C_n \ldots\ldots\ldots (3),$$

where A_n, B_n, C_n are spherical surface-harmonics of order n.

We seek a solution of equations (1) expressed in terms of spherical harmonics.

Differentiate equations (1) with respect to x, y, z, add, and use (2), and we find

$$(\lambda+2\mu)\nabla^2\Delta=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Thus Δ satisfies Laplace's equation, and therefore, within a sphere whose centre is the origin, Δ may be expanded in a series of spherical solid harmonics, so that we may write

$$\Delta=\sum_{n=0}^{n=\infty}\theta_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where θ_n is a spherical solid harmonic of order n.

Now $\partial\theta_n/\partial x$ is a spherical solid harmonic of order $(n-1)$, and thus

$$\nabla^2\left(r^2\frac{\partial\theta_n}{\partial x}\right) = 2(2n+1)\frac{\partial\theta_n}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where r is the distance of the point (x, y, z) from the origin. Thus we get a particular solution of the equation

$$\mu\nabla^2 u = -(\lambda+\mu)\frac{\partial\Delta}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

in the form

$$u = -\frac{\lambda+\mu}{2\mu}r^2\sum_{n=0}^{n=\infty}\frac{1}{2n+1}\frac{\partial\theta_n}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

and we have similar particular integrals of the equations for v and w.

We have to add to the particular solutions complementary solutions, so arranged that the complete expression of

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$$

may be identical with Δ. Suppose these complementary solutions are ΣU_n, ΣV_n, ΣW_n, where U_n, V_n, W_n are spherical solid harmonics of order n, then we have identically

$$\Sigma\left[\frac{\partial U_{n+1}}{\partial x}+\frac{\partial V_{n+1}}{\partial y}+\frac{\partial W_{n+1}}{\partial z}\right]-\frac{\lambda+\mu}{\mu}\Sigma\frac{n}{2n+1}\theta_n = \Sigma\theta_n \ldots(9),$$

where we have picked out the terms containing spherical solid harmonics of the same order n.

Thus, if we write

$$\frac{\partial U_{n+1}}{\partial x}+\frac{\partial V_{n+1}}{\partial y}+\frac{\partial W_{n+1}}{\partial z} = \psi_n \ldots\ldots\ldots\ldots\ldots(10),$$

ψ_n will be a spherical solid harmonic of order n, and

$$\psi_n = \frac{\lambda n+\mu(3n+1)}{\mu(2n+1)}\theta_n \ldots\ldots\ldots\ldots\ldots(11),$$

and the complete expressions for u, v, w are of the forms

$$\left.\begin{aligned} u &= \sum_1^\infty\left(U_n - M_n r^2\frac{\partial\psi_{n-1}}{\partial x}\right),\\ v &= \sum_1^\infty\left(V_n - M_n r^2\frac{\partial\psi_{n-1}}{\partial y}\right),\\ w &= \sum_1^\infty\left(W_n - M_n r^2\frac{\partial\psi_{n-1}}{\partial z}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(12)$$

where $$\psi_{n-1} = \frac{\partial U_n}{\partial x} + \frac{\partial V_n}{\partial y} + \frac{\partial W_n}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots(13),$$

and $$M_n = \tfrac{1}{2} \frac{\lambda + \mu}{\lambda(n-1) + \mu(3n-2)} \ldots\ldots\ldots\ldots\ldots(14).$$

In (12) we have picked out the terms which are homogeneous of the nth degree in (x, y, z), we may also pick out the terms which contain spherical surface-harmonics of order n, and thus write

$$u = \sum_1^\infty \left(U_n - M_{n+2} r^2 \frac{\partial \psi_{n+1}}{\partial x} \right) \quad \ldots\ldots\ldots\ldots\ldots(15),$$

with similar expressions for v and w.

Now, to satisfy the boundary-conditions (3) we have such equations as

$$\Sigma \left[\left(U_n - M_{n+2} a^2 \frac{\partial \psi_{n+1}}{\partial x} \right) - A_n \frac{r^n}{a^n} \right] = 0 \quad \ldots\ldots\ldots(16),$$

when $r = a$.

The left-hand side satisfies Laplace's equation within the sphere of radius a, and vanishes at the surface, it is therefore identically zero. There are three such equations as (16) which are all true identically, and it is clear that the terms of any order n separately vanish for all values of r.

Differentiating equations such as (16) with the sign Σ omitted with respect to x, y, z, and adding, we find

$$\psi_{n-1} = \frac{\partial}{\partial x}\left(A_n \frac{r^n}{a^n} \right) + \frac{\partial}{\partial y}\left(B_n \frac{r^n}{a^n} \right) + \frac{\partial}{\partial z}\left(C_n \frac{r^n}{a^n} \right) \ldots\ldots(17),$$

which determines the function ψ_{n-1}, and in like manner all the functions ψ are determined. Then U_n is determined from (16), and V_n, W_n are given by similar equations. Thus we have finally

$$u = \sum_{n=1}^{n=\infty} \left[A_n \frac{r^n}{a^n} + (a^2 - r^2) M_{n+2} \frac{\partial \psi_{n+1}}{\partial x} \right] \ldots\ldots\ldots(18),$$

with similar expressions for v and w, where

$$\psi_{n+1} = \frac{\partial}{\partial x}\left(A_{n+2} \frac{r^{n+2}}{a^{n+2}} \right) + \frac{\partial}{\partial y}\left(B_{n+2} \frac{r^{n+2}}{a^{n+2}} \right) + \frac{\partial}{\partial z}\left(C_{n+2} \frac{r^{n+2}}{a^{n+2}} \right) \ldots(19),$$

and

$$M_{n+2} = \tfrac{1}{2} \frac{\lambda + \mu}{\lambda(n+1) + \mu(3n+4)} \ldots\ldots\ldots\ldots\ldots(20).$$

166. Displacement in any Solid.

Equations (12) express in terms of rational integral functions of the coordinates quite general solutions of the equations of equilibrium for a simply-connected region containing the origin. For such a region Δ can always be expressed in a series of spherical solid harmonics such as θ_n, and the displacements consist of particular integrals of the differential equations of the form given by (8) and complementary solutions of the same equations expressible in a series of spherical solid harmonics, and the four sets of harmonics thus introduced are connected by the set of relations involved in the equation $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = \Delta$. Mr Chree[1] has applied this method to the determination of general solutions expressed in positive integral powers of the coordinates, and has obtained by this means the displacements in a rotating ellipsoid. He has also shewn that Saint-Venant's solution of the problem of the flexure of an elliptic beam is the only possible solution which contains no higher power than the third of the coordinates of a point on the cross-section.

167. The sphere with given surface-tractions.

Suppose that, at $r = a$, the surface-tractions F, G, H are given. We may suppose them expressed in spherical harmonic series in the forms

$$F = \sum_1^\infty F_n, \quad G = \sum_1^\infty G_n, \quad H = \sum_1^\infty H_n \ldots\ldots\ldots\ldots (21),$$

where F_n, G_n, H_n are spherical surface-harmonics of order n.

Now the boundary-conditions are three such equations as

$$F = \frac{x}{r}\lambda\Delta + 2\mu\frac{x}{r}\frac{\partial u}{\partial x} + \mu\frac{y}{r}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu\frac{z}{r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \ldots (22)$$

when $r = a$, and these are equivalent to three such as

$$Fr = \lambda x\Delta + \mu\left(r\frac{\partial u}{\partial r} + \frac{\partial \zeta}{\partial x} - u\right) \ldots\ldots\ldots\ldots (23),$$

where

$$\zeta = ux + vy + wz \ldots\ldots\ldots\ldots (24),$$

so that ζ/r is the radial displacement.

Suppose u, v, w found so as to satisfy the differential equations

[1] *Quarterly Journal*, 1886 and 1888.

and the boundary-conditions, and their surface-values expressed in spherical surface-harmonics, then we shall have, when $r = a$,

$$u = \Sigma A_n, \quad v = \Sigma B_n, \quad w = \Sigma C_n,$$

where A_n, B_n, C_n are spherical surface-harmonics, and we know that at all internal points u, v, w can be expressed in such forms as

$$u = \Sigma \left(A_n \frac{r^n}{a^n} + a^2 M_{n+2} \frac{\partial \psi_{n+1}}{\partial x} - r^2 M_n \frac{\partial \psi_{n-1}}{\partial x} \right) \ldots\ldots(25),$$

where

$$\psi_{n-1} = \frac{\partial}{\partial x}\left(A_n \frac{r^n}{a^n} \right) + \frac{\partial}{\partial y}\left(B_n \frac{r^n}{a^n} \right) + \frac{\partial}{\partial z}\left(C_n \frac{r^n}{a^n} \right),$$

$$M_n = \tfrac{1}{2} \frac{\lambda + \mu}{\lambda (n-1) + \mu (3n-2)}.$$

Thus if A_n, B_n, C_n could be expressed in terms of the surface-tractions the problem would be solved.

We have to calculate the surface-tractions corresponding to displacements such as (25).

For this purpose we first write down the value of Δ; it is

$$\Delta = \Sigma \frac{2\mu (2n-1)}{\lambda + \mu} M_n \psi_{n-1}.$$

Then we transform $x\Delta$ by the aid of the identity

$$xf(x, y, z) = \frac{r^2}{2n+1} \left[\frac{\partial f}{\partial x} - \frac{r^{2n+1}}{a^{2n+1}} \frac{\partial}{\partial x} \left(\frac{a^{2n+1}}{r^{2n+1}} f \right) \right] \ldots\ldots(26).$$

We thus find (writing $n-1$ for n and ψ_{n-1} for f)

$$x\Delta = r^2 \frac{2\mu}{\lambda + \mu} \Sigma \left[M_n \left\{ \frac{\partial \psi_{n-1}}{\partial x} - \frac{r^{2n-1}}{a^{2n-1}} \frac{\partial}{\partial x} \left(\frac{a^{2n-1}}{r^{2n-1}} \psi_{n-1} \right) \right\} \right] \ldots(27).$$

We now introduce a new function ϕ_{-n-2}, which is a spherical solid harmonic of negative order $-(n+2)$ defined by the equation

$$\phi_{-n-2} = \frac{\partial}{\partial x}\left(A_n \frac{a^{n+1}}{r^{n+1}} \right) + \frac{\partial}{\partial y}\left(B_n \frac{a^{n+1}}{r^{n+1}} \right) + \frac{\partial}{\partial z}\left(C_n \frac{a^{n+1}}{r^{n+1}} \right) \ldots(28),$$

and use the identity (26) to transform the expression

$$(xA_n + yB_n + zC_n) \frac{r^n}{a^n}.$$

We find

$$(xA_n + yB_n + zC_n) \frac{r^n}{a^n} = \frac{r^2}{2n+1} \left(\psi_{n-1} - \frac{r^{2n+1}}{a^{2n+1}} \phi_{-n-2} \right).$$

Thus we obtain

$$\zeta = \Sigma \left[\frac{r^2}{2n+1} \left(\psi_{n-1} - \frac{r^{2n+1}}{a^{2n+1}} \phi_{-n-2} \right) + M_{n+2} a^2 (n+1) \psi_{n+1} - M_n r^2 (n-1) \psi_{n-1} \right] \quad \ldots\ldots(29),$$

where the terms expressed are homogeneous of the $(n+1)$th degree in x, y, z.

Hence we find

$$\frac{\partial \zeta}{\partial x} = \Sigma \left[M_{n+2} a^2 (n+1) \frac{\partial \psi_{n+1}}{\partial x} - \frac{1}{2n+1} \frac{\partial}{\partial x} \left(\frac{r^{2n+3}}{a^{2n+1}} \phi_{-n-2} \right) + r^2 \left\{ \frac{1}{2n+1} - (n-1) M_n \right\} \frac{\partial \psi_{n-1}}{\partial x} + \frac{2r^2}{2n-1} \left\{ \frac{1}{2n+1} - (n-1) M_n \right\} \left\{ \frac{\partial \psi_{n-1}}{\partial x} - \frac{r^{2n-1}}{a^{2n-1}} \frac{\partial}{\partial x} \left(\frac{a^{2n-1}}{r^{2n-1}} \psi_{n-1} \right) \right\} \right] \quad \ldots\ldots\ldots(30),$$

where we have used an identity similar to (26) to simplify $x\psi_{n-1}$, and have picked out the terms of degree n.

Also we find easily

$$r \frac{\partial u}{\partial r} - u = \Sigma \left[(n-1) \left(A_n \frac{r^n}{a^n} + a^2 M_{n+2} \frac{\partial \psi_{n+1}}{\partial x} - r^2 M_n \frac{\partial \psi_{n-1}}{\partial x} \right) \right] \quad (31).$$

Hence the terms of degree n in Fr/μ are

$$(n-1) A_n \frac{r^n}{a^n} + 2n M_{n+2} a^2 \frac{\partial \psi_{n+1}}{\partial x} - 2(n-2) M_n r^2 \frac{\partial \psi_{n-1}}{\partial x} - \frac{1}{2n+1} \frac{\partial}{\partial x} \left(\frac{r^{2n+3}}{a^{2n+1}} \phi_{-n-2} \right) - E_n \frac{r^{2n+1}}{a^{2n-1}} \frac{\partial}{\partial x} \left(\frac{a^{2n-1}}{r^{2n-1}} \psi_{n-1} \right) \ldots(32),$$

where $$E_n = \frac{1}{2n+1} \frac{\lambda(n+2) - \mu(n-3)}{\lambda(n-1) + \mu(3n-2)} \quad \ldots\ldots\ldots\ldots(33).$$

Rearranging, and picking out terms that contain surface-harmonics of order n, we have the surface-tractions required to produce surface-displacements $A_n, \ldots$ expressed in terms of the given surface-tractions $F_n, \ldots$ by equations of the form

$$(n-1) A_n \frac{r^n}{a^n} - E_n \frac{r^{2n+1}}{a^{2n-1}} \frac{\partial}{\partial x} \left(\frac{a^{2n-1}}{r^{2n-1}} \psi_{n-1} \right) - \frac{1}{2n+1} \frac{\partial}{\partial x} \left(\frac{r^{2n+3}}{a^{2n+1}} \phi_{-n-2} \right) - \frac{F_n r^n}{\mu a^{n-1}} = 0 \quad \ldots\ldots\ldots\ldots(34),$$

when $r = a$.

The left-hand side of (34) satisfies Laplace's equation within the sphere of radius a, and vanishes at the surface, it is therefore identically zero. We have three such equations as (34) which are all true identically, and these hold for each value of n and for all values of r.

Introduce two new functions Ψ_{n-1}, Φ_{-n-2}, which are spherical solid harmonics of orders indicated by their suffixes, and defined by the equations

$$\left.\begin{aligned} \Psi_{n-1} &= \frac{\partial}{\partial x}\left(\frac{r^n}{a^n}F_n\right)+\frac{\partial}{\partial y}\left(\frac{r^n}{a^n}G_n\right)+\frac{\partial}{\partial z}\left(\frac{r^n}{a^n}H_n\right), \\ \Phi_{-n-2} &= \frac{\partial}{\partial x}\left(\frac{a^{n+1}}{r^{n+1}}F_n\right)+\frac{\partial}{\partial y}\left(\frac{a^{n+1}}{r^{n+1}}G_n\right)+\frac{\partial}{\partial z}\left(\frac{a^{n+1}}{r^{n+1}}H_n\right) \end{aligned}\right\} \quad \ldots(35).$$

Take equation (34) and the two similar equations, differentiate them with respect to x, y, z, add, and we get

$$\psi_{n-1}=\frac{\lambda(n-1)+\mu(3n-2)}{\lambda(2n^2+1)+2\mu(n^2-n+1)}\frac{a}{\mu}\Psi_{n-1} \quad \ldots\ldots(36).$$

Take the same equations, multiply them by x, y, z, add, and use identity (26) and equation (36), and we get

$$\phi_{-n-2}=\frac{a}{2n\mu}\Phi_{-n-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(37).$$

Thus the functions ψ_{n-1} and ϕ_{-n-2}, defined in equations (17) and (28) in terms of surface-displacements, are definite multiples of the corresponding functions Ψ_{n-1} and Φ_{-n-2}, defined in like manner in terms of surface-tractions, and are, therefore, completely determined. Hence, by equation (34), A_n is completely determined, and B_n and C_n are given by like equations. Finally u, v, w are given by (25) and the similar equations in terms of the A_n, B_n, C_n, ψ_{n-1}, and thus u, v, w are completely determined.

The possible values of n must be chosen, so that the system of applied surface-tractions may satisfy the conditions of equilibrium of a system of forces applied to a rigid body. It is easy to shew that $n=0$ does not occur, and that, if $n=1$, rF_1, rG_1, rH_1 are the partial differential coefficients with respect to x, y, z of a homogeneous quadratic function of these variables. In the general problem of art. 170 these conditions are satisfied identically.

168. Case of purely normal surface-tractions.

Suppose the surface-tractions equivalent to a normal traction equal at any point of the surface to

$$\sum_1^\infty R_n,$$

where R_n is a spherical surface-harmonic of order n; then we must have when $r = a$

$$Fr = x\sum_1^\infty R_n\left(\frac{r}{a}\right)^n, \quad Gr = y\sum_1^\infty R_n\left(\frac{r}{a}\right)^n, \quad Hr = z\sum_1^\infty R_n\left(\frac{r}{a}\right)^n \ldots(38).$$

Now the surface-tractions that must be applied to the surface $r = a$ to produce the system of displacements (25) are given by (34), and may be written in forms of which the type is given by

$$Fr = \Sigma\left[\mu(n-1)A_n\frac{r^n}{a^n} - \frac{\mu}{2n+1}\frac{\partial}{\partial x}\left(\frac{r^{2n+3}}{a^{2n+1}}\phi_{-n-2}\right)\right.$$
$$\left. - \mu E_{n+2}r^{2n+5}\frac{\partial}{\partial x}\left(\frac{\psi_{n+1}}{r^{2n+3}}\right)\right] \ldots\ldots\ldots(39).$$

The surface-tractions actually applied can be written in forms of which the type is

$$Fr = \Sigma\frac{1}{2n+3}\left[a^2\frac{\partial}{\partial x}\left(R_{n+1}\frac{r^{n+1}}{a^{n+1}}\right) - \frac{r^{2n+5}}{a^{2n+3}}\frac{\partial}{\partial x}\left(R_{n+1}\frac{a^{n+2}}{r^{n+2}}\right)\right] \ldots(40).$$

If we equate the right-hand sides of (39) and (40) it is easy to shew, as in the last article, that we have an equation which is true identically; and we can obtain, by considering the expressions for Gr and Hr, two other like identities. We first differentiate these with respect to x, y, z and add; then we multiply them by x, y, z and add. We thus get the two equations

$$\Sigma\mu\left[(n+1)+(n+2)(2n+5)E_{n+2}\right]\psi_{n+1}$$
$$= \Sigma\frac{(n+2)(2n+5)}{2n+3}\left(\frac{r}{a}\right)^{2n+3}\left(R_{n+1}\frac{a^{n+2}}{r^{n+2}}\right),$$

and

$$\Sigma\mu r^2\left[(n+1)+(n+2)(2n+5)E_{n+2}\right]\frac{\psi_{n+1}}{2n+5} - \Sigma\mu\frac{2n}{2n+1}\frac{r^{2n+3}}{a^{2n+1}}\phi_{-n-2}$$
$$= \Sigma\left[\frac{n+1}{2n+3}a^2\left(R_{n+1}\frac{r^{n+1}}{a^{n+1}}\right) + \frac{(n+2)r^{2n+5}}{(2n+3)a^{2n+3}}R_{n+1}\frac{a^{n+2}}{r^{n+2}}\right].$$

At the surface the first of these gives us

$$\mu\left[(n+1)+(n+2)(2n+5)E_{n+2}\right]\psi_{n+1} = \frac{(n+2)(2n+5)}{2n+3}\left(\frac{r}{a}\right)^{n+1}R_{n+1}\ldots\ldots(41)$$

and, using this to simplify the second, we get at the surface

$$-\mu\frac{2n}{2n+1}\left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2} = \frac{n+1}{2n+3}\left(\frac{r}{a}\right)^{n+1}R_{n+1}\ldots\ldots(42),$$

where we have picked out the terms containing surface-harmonics of the $(n+1)$th order. As the equations obtained are relations between the surface-values of spherical solid harmonics of the same order they hold throughout the sphere.

To determine the A's, B's, and C's we shall suppose the surface-traction expressed by a single term R_{n+1}. In this case ψ_{n+1} and ϕ_{-n-2} are the only functions ψ and ϕ that occur. Equating (39) and (40), and picking out terms which contain surface-harmonics of order n and of order $n+2$, we get

$$\mu(n-1)A_n\frac{r^n}{a^n} = a^2\left[\frac{\mu}{2n+1}\frac{\partial}{\partial x}\left(\frac{r^{2n+3}}{a^{2n+3}}\phi_{-n-2}\right)+\frac{1}{2n+3}\frac{\partial}{\partial x}\left(R_{n+1}\frac{r^{n+1}}{a^{n+1}}\right)\right],$$

and

$$\mu(n+1)A_{n+2}\frac{r^{n+2}}{a^{n+2}} = \mu E_{n+2}r^{2n+5}\frac{\partial}{\partial x}\left(\frac{\psi_{n+1}}{r^{2n+3}}\right)-\frac{1}{2n+3}\frac{r^{2n+5}}{a^{2n+3}}\frac{\partial}{\partial x}\left(R_{n+1}\frac{a^{n+2}}{r^{n+2}}\right).$$

Simplifying these by means of (41) and (42), we find

$$\mu A_n\frac{r^n}{a^n} = \frac{1}{2n(2n+3)}\frac{\partial}{\partial x}\left(\frac{r^{n+1}}{a^{n-1}}R_{n+1}\right)\ldots\ldots\ldots\ldots(43)$$

and

$$-\mu A_{n+2}\frac{r^{n+2}}{a^{n+2}} = \frac{1}{n+1+(n+2)(2n+5)E_{n+2}}\cdot\frac{r^{2n+5}}{(2n+3)a^{2n+3}}\frac{\partial}{\partial x}\left(\frac{a^{n+2}}{r^{n+2}}R_{n+1}\right)\ldots(44).$$

These are the only A's that occur, and the B's and C's can be found by writing y and z respectively for x.

169. Spherical Cavity in infinite solid mass[1].

By analogy with the solutions for space inside a sphere we may write down those for space outside in forms of which the type is

$$u = \Sigma \left\{ \frac{U_n}{r^{2n+1}} - k_n r^2 \frac{\partial}{\partial x} \left(\frac{\psi_{n+1}}{r^{2n+3}} \right) \right\} \quad \ldots\ldots\ldots\ldots\ldots (45),$$

where v and w are found by cyclical interchanges of the letters (U, V, W), (x, y, z), and ψ_{n+1} is given by the equation

$$\psi_{n+1} = r^{2n+3} \left\{ \frac{\partial}{\partial x} \left(\frac{U_n}{r^{2n+1}} \right) + \frac{\partial}{\partial y} \left(\frac{V_n}{r^{2n+1}} \right) + \frac{\partial}{\partial z} \left(\frac{W_n}{r^{2n+1}} \right) \right\} \ldots\ldots (46).$$

Also U_n, V_n, W_n are spherical solid harmonics of positive degree n, and therefore ψ_{n+1} is a spherical solid harmonic of degree $n+1$.

These forms satisfy the differential equations of equilibrium (1) if

$$k_n = -\tfrac{1}{2} \frac{\lambda + \mu}{(n+2)\lambda + (3n+5)\mu} \quad \ldots\ldots\ldots\ldots\ldots (47),$$

and it can be shewn that the cubical dilatation Δ is given by the equation

$$\Delta = -2\Sigma \frac{(2n+3)\mu}{\lambda + \mu} k_n \frac{\psi_{n+1}}{r^{2n+3}} \quad \ldots\ldots\ldots\ldots\ldots (48).$$

It can be shewn in the manner of art. 167 that the surface-tractions F, G, H at the surface of the cavity $r = a$, required to maintain the state of strain expressed by such equations as (45), are given by formulæ of which the type is

$$-\frac{Fr}{\mu} = \Sigma \left[P_n \frac{r^{2n+5}}{a^{2n+5}} \frac{\partial}{\partial x} \left(\frac{\psi_{n+1}}{r^{2n+3}} \right) + Q_n \frac{\partial \psi_{n+1}}{\partial x} + P_n' \frac{r^{2n+1}}{a^{2n+1}} \frac{\partial}{\partial x} \left(\frac{\chi_{n-1}}{r^{2n-1}} \right) - (n+2) \frac{U_n}{a^{2n+1}} \right] \quad \ldots\ldots\ldots\ldots (49),$$

in which

$$\chi_{n-1} = \frac{\partial U_n}{\partial x} + \frac{\partial V_n}{\partial y} + \frac{\partial W_n}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots (50),$$

and

$$\left.\begin{aligned} P_n &= a^2 k_n \left\{ \frac{2\lambda}{\lambda + \mu} + 2(n+2) \frac{n+1}{2n+3} \right\} - \frac{a^2}{2n+3}, \\ Q_n &= \frac{k_n}{a^{2n+1}} \left\{ \frac{2(n+2)}{2n+3} - \frac{2\lambda}{\lambda + \mu} \right\} - \frac{2a^{-2n-1}}{(2n+1)(2n+3)}, \\ P_n' &= \frac{1}{2n+1} \end{aligned}\right\} (51).$$

From the equations of type (49), it is easy, by the method of art. 167, to express ψ_{n+1} and χ_{n-1} in terms of given surface-

[1] Only the leading steps of the analysis are given.

tractions, and thence to infer the values of U_n, V_n, W_n. We shall consider a particular example.

Suppose that at a very great distance there is a finite shear s, so that the displacement, when $r = \infty$, can be expressed by

$$u = sy, \quad v = 0, \quad w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (52).$$

In this case, it can be shewn, by the above analysis, that, if the surface $r = a$ be free, the only harmonics ψ and χ that occur are proportional to xy, and that the only harmonics U that occur are U_1 and U_3; also that U_1 and V_1 are respectively proportional to y and x, while $W_1 = 0$, and that U_3 is proportional to $r^7 \dfrac{\partial}{\partial x}\left(\dfrac{xy}{r^5}\right)$, and $V_3 r^{-7}$, $W_3 r^{-7}$ are the same multiples of the differential coefficients of xyr^{-5} with respect to y and z.

We thus find for the forms of u, v, w

$$\left.\begin{aligned} u &= A\frac{y}{r^3} + B\frac{\partial}{\partial x}\left(\frac{xy}{r^5}\right) + Cr^2\frac{\partial}{\partial x}\left(\frac{xy}{r^5}\right) + sy, \\ v &= A\frac{x}{r^3} + B\frac{\partial}{\partial y}\left(\frac{xy}{r^5}\right) + Cr^2\frac{\partial}{\partial y}\left(\frac{xy}{r^5}\right), \\ w &= \quad B\frac{\partial}{\partial z}\left(\frac{xy}{r^5}\right) + Cr^2\frac{\partial}{\partial z}\left(\frac{xy}{r^5}\right) \end{aligned}\right\} \ldots\ldots (53),$$

where A, B, C are constants.

If the spherical cavity be free from stress, the constants A, B, C are given by the equations[1]

$$\left.\begin{aligned} A &= \frac{3\lambda + 8\mu}{9\lambda + 14\mu} a^3 s, \\ B &= \frac{3(\lambda + \mu)}{9\lambda + 14\mu} a^5 s \\ C &= -\frac{3(\lambda + \mu)}{9\lambda + 14\mu} a^3 s \end{aligned}\right\} \ldots\ldots\ldots\ldots (54).$$

The most important point concerns the shear $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$. It is easy to shew that the value of this, when $x = 0$, $y = 0$, $r = a$, is

$$\frac{15\lambda + 30\mu}{9\lambda + 14\mu} s \ldots\ldots\ldots\ldots\ldots\ldots (55),$$

[1] See *Phil. Mag.* Jan. 1892, p. 77.

so that if $\lambda = \mu$ (Poisson's condition), the value of the shear near the cavity may be very little less than twice that at a considerable distance. This shews that the existence of a flaw[1], in the form of a spherical cavity, may cause a serious diminution of strength in a body subjected to shearing forces.

170. General Problem.

We shall now consider the following problem:

A solid isotropic homogeneous elastic sphere, formed of gravitating matter, is rotating slowly about a diameter, and is subject to the action of forces derivable from a potential expressible in spherical harmonic series—it is required to find the state of strain in the interior.

Suppose a' is the unstrained radius of the sphere. Then, as part of the bodily force is radial, the sphere will be compressed, and the equation of the surface of the strained sphere will be of the form

$$r = a + \Sigma \epsilon_n Q_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56),$$

where a is slightly different from a', Q_{n+1} denotes a spherical solid harmonic of order $n+1$, and ϵ_n is a small constant coefficient.

We next consider the composition of the bodily forces. In the first place, owing to the rotation, we shall have to include terms depending on "centrifugal force". These may be regarded as derivable from a potential $\frac{1}{3}\omega^2 r^2 (1 - P_2)$, where P_2 is Legendre's second coefficient, and ω is the angular velocity of rotation about the axis z. Suppose now that g is the value of gravity at the mean strained surface $r = a$, and that $\omega^2 a/g$ is a small quantity which may be neglected if it occurs as a factor in a product of which some other factor is considered small. Then the bodily forces consist of three sets:

(i) Radial forces $-gr/a + \frac{2}{3}\omega^2 r$.

(ii) External forces derivable from a potential expressible in the form ΣW_{n+1} where W_{n+1} is a solid harmonic of order $n+1$. This includes the terms $-\frac{1}{3}\omega^2 r^2 P_2$, arising from the "centrifugal force".

(iii) Forces arising from the attraction of the harmonic inequalities and derivable from a potential which is $\Sigma \dfrac{3g}{2n+3} \epsilon_n Q_{n+1}$.

[1] Cf. arts. 95, 96.

Let us write $$Y_{n+1} = W_{n+1} + \frac{3g}{2n+3}\epsilon_n Q_{n+1} \quad \text{.....(57),}$$

and $$V = \tfrac{1}{2}\left(-\frac{g}{a} + \tfrac{2}{3}\omega^2\right) r^2 + \Sigma Y_{n+1} \quad \text{..... (58).}$$

Then the differential equations are three such as

$$(\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 u + \rho \frac{\partial V}{\partial x} = 0 \quad \text{..... (59).}$$

We have to find values of u, v, w from these which are finite and continuous within the surface $r = a + \Sigma \epsilon_n Q_{n+1}$, and satisfy the condition that this surface is free from stress.

The solution consists of two parts. We have first to find any particular solutions of the above system of equations. The particular solutions that we can most easily find do not satisfy the boundary-condition. We must therefore add to them complementary solutions of the equations that would be derived by putting V equal to zero. When the complementary solutions are properly chosen the boundary-condition will be satisfied.

Now, as V is a sum of terms, and the equations linear, it is clear that we may find the particular solutions corresponding to each term separately and add the results. We shall therefore consider V to consist of two terms, viz.: $\frac{1}{2}(\frac{2}{3}\omega^2 - g/a) r^2$ and Y_{n+1}. Our process consists in finding separately the particular integrals depending on these two terms, and the terms contributed by these particular integrals to the tractions at the surface $r = a + \Sigma \epsilon_n Q_{n+1}$. We shall assume that terms in which such products as $\epsilon_m Q_{m+1} Y_{n+1}$ or $\epsilon_n \epsilon_m Q_{n+1} Q_{m+1}$ occur may be neglected, also we shall neglect the product $\omega^2 \epsilon_n$, but we shall suppose that $g\epsilon_n$ is a quantity of the order retained. This is equivalent to supposing that the strains produced by gravitation are large compared with those produced by the external disturbing force or the "centrifugal force". It does not require us to suppose the gravitational strain large enough to be necessarily accompanied by permanent set.

171. Particular Integral for the Radial Forces.

We have given in art. 127 the solution of the differential equations of displacement due to radial forces proportional to r. We found that they were satisfied by supposing the displacement purely radial and equal to $Ar + Hr^3$, where A is an arbitrary constant and H is a given constant.

We therefore have particular solutions of the equations such as

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u+\rho\frac{\partial}{\partial x}\left\{\tfrac{1}{2}r^2\left(\tfrac{2}{3}\omega^2-\frac{g}{a}\right)\right\}=0\ldots\ldots(60)$$

in the forms

$$u=\frac{x}{r}(Ar+Hr^3),\quad v=\frac{y}{r}(Ar+Hr^3),\quad w=\frac{z}{r}(Ar+Hr^3)\ldots(61),$$

where

$$H=\tfrac{1}{10}\left(\frac{g}{a}-\tfrac{2}{3}\omega^2\right)\frac{\rho}{\lambda+2\mu}\ldots\ldots\ldots\ldots\ldots\ldots(62).$$

We found in the same article that the traction across a sphere of radius r is radial and equal to

$$(3\lambda+2\mu)A+(5\lambda+6\mu)Hr^2\ldots\ldots\ldots\ldots\ (63).$$

It follows from this that the surface-tractions F, G, H across the surface $r=a+\Sigma\epsilon_n Q_{n+1}$ will contain terms such as

$$\frac{x}{a}\{(3\lambda+2\mu)A+(5\lambda+6\mu)Ha^2\},$$

as well as terms depending on the spherical harmonics.

All the terms contributed to the surface-tractions by the particular integrals depending on Y_{n+1}, and by the complementary solutions, will contain spherical harmonics like Y_{n+1} or Q_{n+1}, and thus the terms found above will have to vanish.

This finds the same value for A as that given in art. 127, viz.:

$$A=-\frac{5\lambda+6\mu}{3\lambda+2\mu}Ha^2\ldots\ldots\ldots\ldots\ldots\ldots(64).$$

In what follows we shall suppose A to have this value.

172. Surface-tractions depending on Radial Forces.

These arise from the displacements

$$u=Ax+Hr^2x,\quad v=Ay+Hr^2y,\quad w=Az+Hr^2z.$$

The six strains e, f, g, a, b, c are given by such equations as

$$e=A+H(r^2+2x^2),\ldots a=4Hyz,\ldots$$

and the cubical dilatation Δ is given by the equation

$$\Delta=3A+5Hr^2.$$

The six stresses P, Q, R, S, T, U, hence arising, are given by such equations as

$$P=\lambda\Delta+2\mu e,\ldots S=\mu a,\ldots$$

Now the direction-cosines l', m', n' of the outward-drawn normal to the surface

$$r = a + \epsilon_n Q_{n+1},$$

where Q_{n+1} is a spherical solid harmonic, and ϵ_n is small, are given by such equations as

$$l' = \frac{x}{r} + \epsilon_n \left\{ \frac{(n+1)\,x}{r^2} Q_{n+1} - \frac{\partial Q_{n+1}}{\partial x} \right\} \quad \ldots\ldots\ldots(65),$$

and m', n' are similar expressions.

The surface-tractions at the deformed surface that arise from the purely radial forces are three such expressions as

$$l'P + m'U + n'T.$$

Now

$$P\frac{x}{r} + U\frac{y}{r} + T\frac{z}{r} = [Hr^2(5\lambda + 6\mu) + A(3\lambda + 2\mu)]\frac{x}{r} \ldots(66),$$

and

$$P\frac{\partial Q_{n+1}}{\partial x} + U\frac{\partial Q_{n+1}}{\partial y} + T\frac{\partial Q_{n+1}}{\partial z}$$
$$= \left[Hr^2(5\lambda + 2\mu)\frac{\partial Q_{n+1}}{\partial x} + 4\mu H(n+1)\,xQ_{n+1} + (3\lambda + 2\mu)\,A\frac{\partial Q_{n+1}}{\partial x} \right] \quad \ldots\ldots\ldots(67).$$

Thus the part contributed to Fr is

$$x[Ha^2(5\lambda + 6\mu) + A(3\lambda + 2\mu)]\left[1 + \frac{(n+1)}{a}\epsilon_n Q_{n+1}\right]$$
$$+ 2Ha\epsilon_n Q_{n+1}(5\lambda + 6\mu)\,x - a\epsilon_n \left[Ha^2(5\lambda + 2\mu)\frac{\partial Q_{n+1}}{\partial x} \right.$$
$$\left. + 4\mu H(n+1)\,xQ_{n+1} + (3\lambda + 2\mu)\,A\frac{\partial Q_{n+1}}{\partial x} \right],$$

of which the first term vanishes identically by (64), and the second is obtained by substituting for r its value $a + \epsilon_n Q_{n+1}$ in (66).

Thus, collecting the terms in xQ_{n+1} and $\dfrac{\partial Q_{n+1}}{\partial x}$, and transforming by means of the identity (26), we have for the part contributed to Fr by the strain produced by the radial forces

$$Ha\epsilon_n \left[2\frac{5\lambda - (2n-4)\,\mu}{2n+3} r^2 \frac{\partial Q_{n+1}}{\partial x} + 4\mu a^2 \frac{\partial Q_{n+1}}{\partial x} \right.$$
$$\left. - 2\frac{5\lambda - (2n-4)\,\mu}{2n+3} r^{2n+5} \frac{\partial}{\partial x}\left(\frac{Q_{n+1}}{r^{2n+3}}\right) \right] \quad \ldots\ldots\ldots(68).$$

This is the typical term arising from the spherical harmonic term $\epsilon_n Q_{n+1}$ in the equation of the surface.

Since we neglect ϵ_n^2, $\epsilon_n \epsilon_m$, and $\epsilon_n \omega^2$, we may take H to be $\frac{1}{10} \frac{g\rho}{(\lambda + 2\mu)\, a}$, and write $a^2 \frac{\partial Q_{n+1}}{\partial x}$ for $r^2 \frac{\partial Q_{n+1}}{\partial x}$ in the surface-value of (68), so that this surface-value becomes

$$\tfrac{1}{5} \frac{g\rho}{\lambda + 2\mu} \Sigma \epsilon_n \left[\frac{5\lambda + 2(n+5)\mu}{2n+3} a^2 \frac{\partial Q_{n+1}}{\partial x} - \frac{5\lambda - 2(n-2)\mu}{2n+3} r^{2n+5} \frac{\partial}{\partial x} \left(\frac{Q_{n+1}}{r^{2n+3}} \right) \right] \quad \ldots\ldots(69).$$

This is the part contributed to the value of Fr at the free surface by the strain produced by the radial forces.

173. Digression on certain tractions.

The formula we have just obtained is very important. To see its meaning we may with advantage consider particular cases.

Take first the case where the solid is incompressible. In this case λ is infinite, and the formula may be written

$$g\rho \Sigma \epsilon_n \frac{r^2}{2n+3} \left[\frac{\partial Q_{n+1}}{\partial x} - r^{2n+3} \frac{\partial}{\partial x} \left(\frac{Q_{n+1}}{r^{2n+3}} \right) \right]$$

and this is, by (26),

$$g\rho x \Sigma \epsilon_n Q_{n+1}.$$

Thus the traction in question is a radial traction equal to the weight of the harmonic inequality.

In general the normal traction on the surface of the mean sphere is

$$\tfrac{1}{5} g\rho \Sigma \frac{\{5\lambda + 2(n+5)\mu\}(n+1) + \{5\lambda - 2(n-2)\mu\}(n+2)}{(2n+3)(\lambda + 2\mu)} \epsilon_n Q_{n+1},$$

which is equal to

$$\tfrac{1}{5} g\rho \frac{5\lambda + 6\mu}{\lambda + 2\mu} \Sigma \epsilon_n Q_{n+1} \quad \ldots\ldots(70),$$

so that the normal traction is equal to $\frac{1}{5}(5\lambda + 6\mu)/(\lambda + 2\mu)$ times the weight of the harmonic inequality, and there are also tangential tractions.

According to (70) the x-component of the normal traction is

$$\tfrac{1}{5} g\rho \frac{5\lambda + 6\mu}{\lambda + 2\mu} \frac{x}{r} \Sigma \epsilon_n Q_{n+1},$$

and thus the terms of Fr contributed by the tangential traction are

$$\tfrac{1}{5}\frac{g\rho}{\lambda+2\mu}\Sigma\epsilon_n\left[\frac{2(n+2)\mu}{2n+3}a^2\frac{\partial Q_{n+1}}{\partial x}+\frac{2(n+1)\mu}{2n+3}r^{2n+5}\frac{\partial}{\partial x}\left(\frac{Q_{n+1}}{r^{2n+3}}\right)\right].$$

That the traction thus given is really tangential admits of immediate verification.

In the theory of Hydrostatics we have to consider the effects of harmonic disturbing forces upon a sphere of gravitating incompressible fluid, and it is always supposed that there is a pressure at the mean surface equal to the weight of the harmonic inequality. In like manner in the case of an *incompressible* solid sphere which is elastic in opposing change of shape, and subject to the mutual gravitation of its parts, some writers have supposed that there will be such a pressure on the mean sphere. This supposition finds here its justification.

If we begin with a sphere of radius a, and deform it into an oblate spheroid by paring down the parts near the poles, and adding mass near the equator, it is clear that there must be tractions across the mean sphere to support the weight of the added mass.

In the case of an elastic solid mass we now see that the corresponding traction is not in general normal, nor is its normal component equal to the weight of the harmonic inequality. If we cut out a small part of the harmonic inequality by planes through the centre of the sphere, the weight of the part cut out will be partly supported by the normal pressure on its base and partly by the tangential stresses on its sides. The existence of such tangential stresses involves, according to Cauchy's theorem (art. 14), the existence of tangential stresses in the tangent plane to the mean sphere.

174. Particular Integral for the Disturbing Forces.

Returning to the problem stated in art. 170, we have next to find a particular integral of the equations such as

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u+\rho\frac{\partial Y_{n+1}}{\partial x}=0 \quad \text{.....(71),}$$

where Y_{n+1} is a spherical solid harmonic of order $(n+1)$, with a small coefficient of the same order as ϵ_n.

Now such a particular integral can be found by assuming that

the strain throughout the sphere is irrotational, *i.e.* that there is a displacement-potential ϕ such that

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z},$$

for then $\Delta = \nabla^2 \phi$, and the equations can be satisfied if

$$(\lambda + 2\mu) \nabla^2 \phi + \rho Y_{n+1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (72).$$

Just as in (8) of art. 165 we have a particular integral of this equation in the form

$$\phi = -\frac{\rho}{\lambda + 2\mu} \frac{r^2}{2(2n+5)} Y_{n+1}.$$

Thus the particular integrals u, v, w of equations (71) are given by three such equations as

$$u = -\frac{1}{2(2n+5)} \frac{\rho}{\lambda + 2\mu} \frac{\partial}{\partial x} (r^2 Y_{n+1}) \ldots\ldots\ldots\ldots (73).$$

175. Surface-tractions depending on the particular integrals.

The terms contributed to the cubical dilatation Δ by the particular integrals (73) reduce to

$$-\frac{\rho}{\lambda + 2\mu} Y_{n+1}.$$

The terms contributed to ζ (the product of the radius and the radial displacement) are easily found from (73) to be

$$-\frac{\rho}{\lambda + 2\mu} \frac{n+3}{2(2n+5)} r^2 Y_{n+1}.$$

Thus the terms contributed to Fr by the particular integrals (73) are found by using the formula (23) to be

$$-\frac{\rho}{\lambda + 2\mu} \left[\lambda x Y_{n+1} + \mu \left\{ \frac{n+1}{2(2n+5)} + \frac{n+3}{2(2n+5)} \right\} \frac{\partial}{\partial x} (r^2 Y_{n+1}) \right]$$

and this becomes after differentiation, by using an identity similar to (26) with $(n+1)$ in place of n,

$$-\frac{\rho}{\lambda + 2\mu} \left[\frac{\lambda + \mu(n+2)}{2n+3} r^2 \frac{\partial Y_{n+1}}{\partial x} - \frac{(2n+5)\lambda + 2(n+2)\mu}{(2n+3)(2n+5)} r^{2n+5} \frac{\partial}{\partial x} \left(\frac{Y_{n+1}}{r^{2n+3}} \right) \right] \ldots\ldots\ldots (74).$$

At the free surface we may put a for r after differentiation for the reasons explained in art. 170.

176. The Complementary Solutions.

These can be written in the form given in (18) of art. 165, viz.:

$$u = \Sigma \left[A_n \frac{r^n}{a^n} + (a^2 - r^2) M_{n+2} \frac{\partial \psi_{n+1}}{\partial x} \right];$$

so that the complete expressions for the displacements are three such as

$$\begin{aligned} u = {} & Ax + Hr^2 x \\ & - \frac{\rho}{\lambda + 2\mu} \Sigma \frac{1}{2(2n+5)} \frac{\partial}{\partial x} (r^2 Y_{n+1}) \\ & + \Sigma \left[A_n \frac{r^n}{a^n} + (a^2 - r^2) M_{n+2} \frac{\partial \psi_{n+1}}{\partial x} \right] \end{aligned} \quad \ldots\ldots\ldots\ldots (75),$$

in which

$$\left. \begin{aligned} H = \tfrac{1}{10} \frac{\rho}{\lambda + 2\mu} \left(\frac{g}{a} - \tfrac{2}{3}\omega^2 \right), \quad A = -\frac{5\lambda + 6\mu}{3\lambda + 2\mu} H a^2, \\ M_{n+2} = \tfrac{1}{2} \frac{\lambda + \mu}{\lambda(n+1) + \mu(3n+4)} \end{aligned} \right\} \quad \ldots (76).$$

177. Formation of the boundary-conditions.

Now we may write the expression for the typical terms contributed to the value of Fr when $r = a$ by the complementary functions, as given in (34), in the form

$$\begin{aligned} & \mu(n-1) A_n \frac{r^n}{a^n} - \frac{\mu}{2n+1} \frac{\partial}{\partial x} \left(\frac{r^{2n+3}}{a^{2n+1}} \phi_{-n-2} \right) \\ & \qquad - \mu E_{n+2} \frac{r^{2n+5}}{a^{2n+3}} \frac{\partial}{\partial x} \left(\psi_{n+1} \frac{a^{2n+3}}{r^{2n+3}} \right) \end{aligned} \quad \ldots\ldots\ldots\ldots (77),$$

and the typical terms in the surface-tractions are this and the terms given in (69) and (74). Since the surface is supposed free we must add these terms together and equate the result to zero. We thus obtain an equation which may be written

$$\begin{aligned} & \Sigma \left[a_n \frac{\partial Y_{n+1}}{\partial x} + b_n r^{2n+5} \frac{\partial}{\partial x} \left(\frac{Y_{n+1}}{r^{2n+3}} \right) + a_n' \epsilon_n \frac{\partial Q_{n+1}}{\partial x} + b_n' \epsilon_n r^{2n+5} \frac{\partial}{\partial x} \left(\frac{Q_{n+1}}{r^{2n+3}} \right) \right. \\ & \left. + a_n'' \frac{\partial}{\partial x} \left(\frac{r^{2n+3}}{a^{2n+1}} \phi_{-n-2} \right) + b_n'' r^{2n+5} \frac{\partial}{\partial x} \left(\frac{\psi_{n+1}}{r^{2n+3}} \right) + \mu(n-1) A_n \frac{r^n}{a^n} \right] = 0 \end{aligned}$$

$$\ldots\ldots\ldots\ldots (78),$$

when $r = a$.

The coefficients a_n, b_n,... are

$$\left.\begin{aligned}
a_n &= -\frac{\rho}{\lambda+2\mu}\frac{\lambda+\mu(n+2)}{2n+3}a^2,\\
b_n &= \frac{\rho}{\lambda+2\mu}\frac{(2n+5)\lambda+2(n+2)\mu}{(2n+3)(2n+5)},\\
a_n' &= \tfrac{1}{5}\frac{g\rho}{\lambda+2\mu}\frac{5\lambda+2(n+5)\mu}{2n+3}a^2,\\
b_n' &= -\tfrac{1}{5}\frac{g\rho}{\lambda+2\mu}\frac{5\lambda-2(n-2)\mu}{2n+3},\\
a_n'' &= -\frac{\mu}{2n+1},\\
b_n'' &= -\mu E_{n+2} = -\frac{\mu}{2n+5}\frac{\lambda(n+4)-\mu(n-1)}{\lambda(n+1)+\mu(3n+4)}
\end{aligned}\right\}\quad\ldots(79).$$

Now the left-hand side of (78) is finite, continuous, and one-valued within the sphere $r=a$, satisfies Laplace's equation and vanishes at the surface, it is therefore identically zero for all values of r. We have two other identities of the same form which can be derived from (78) by cyclical interchanges of the letters A, B, C and x, y, z, and the terms of any order n separately vanish.

We can utilise these equations to express the unknown harmonics ψ_{n+1} and ϕ_{-n-2} in terms of Y_{n+1} and Q_{n+1}.

If we differentiate these equations with respect to x, y, z and add, we obtain the equation

$$-(2n+5)(n+2)[b_nY_{n+1}+b_n'\epsilon_nQ_{n+1}+b_n''\psi_{n+1}]+\mu(n+1)\psi_{n+1}=0 \quad\ldots\ldots\ldots(80),$$

where we have picked out the terms which contain surface-harmonics of order $n+1$.

Again if we multiply equation (78) and the like equations by x, y, z, add, and use (26) we get

$$(n+1)\left(a_nY_{n+1}+a_n'\epsilon_nQ_{n+1}+a_n''a^2\left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2}\right) - \mu\frac{(n-1)}{2n+1}a^2\left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2}=0\ldots\ldots\ldots(81),$$

where we have picked out the terms containing surface-harmonics of order $n+1$, and observed that, in virtue of (80), the terms in b_n, b_n', b_n'' and ψ_{n+1} disappear.

The above equations give

$$\left.\begin{aligned}\psi_{n+1}\mu\{(n+1)+(n+2)(2n+5)E_{n+2}\}&\\ =(n+2)(2n+5)(b_nY_{n+1}+b_n'\epsilon_nQ_{n+1})&\\ \left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2}\mu\frac{2n}{2n+1}a^2=(n+1)(a_nY_{n+1}+a_n'\epsilon_nQ_{n+1})&\end{aligned}\right\}$$

.....(82).

In these we can substitute from (57) $W_{n+1}+\frac{3g}{2n+3}\epsilon_nQ_{n+1}$ for Y_{n+1}; and thus we have ψ_{n+1} and ϕ_{-n-2} expressed in terms of W_{n+1} and ϵ_nQ_{n+1}.

To determine Q_{n+1} we remark that, since $r=a+\Sigma\epsilon_nQ_{n+1}$ is the equation of the surface, the radial displacement contains the harmonic terms $\Sigma\epsilon_nQ_{n+1}$ and no others.

Now the radial displacement arising from the particular integrals (61) is

$$Ar+Hr^3.$$

The value of the harmonic terms of this at the surface $r=a+\Sigma\epsilon_nQ_{n+1}$ is

$$\Sigma[A\epsilon_nQ_{n+1}+3Ha^2\epsilon_nQ_{n+1}].$$

The surface-value of the radial displacement arising from the particular integrals (73) is

$$-\Sigma a\frac{\rho}{\lambda+2\mu}\frac{n+3}{2(2n+5)}Y_{n+1}.$$

The surface-value of the radial displacement arising from the complementary functions is by (29)

$$\Sigma\left(\frac{a}{2n+5}\psi_{n+1}-\frac{a}{2n+1}\left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2}\right).$$

Hence equating the sum of these surface-values to $\Sigma\epsilon_nQ_{n+1}$ we get the equation

$$\left.\begin{aligned}a\Sigma\left\{\frac{\psi_{n+1}}{2n+5}-\left(\frac{r}{a}\right)^{2n+3}\frac{\phi_{-n-2}}{2n+1}\right.&\\ \left.-\frac{(n+3)\rho}{2(2n+5)(\lambda+2\mu)}\left(W_{n+1}+\frac{3g}{2n+3}\epsilon_nQ_{n+1}\right)\right\}&\\ =(1-A-3Ha^2)\Sigma\epsilon_nQ_{n+1}&\end{aligned}\right.\quad\text{.....(83),}$$

where we have substituted for Y_{n+1} from (57).

Now ψ_{n+1}, $\left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2}$, and Q_{n+1} are spherical solid harmonics of order $n+1$, and we have obtained in (82) and (83) three equations which determine these in terms of W_{n+1}. It is clear that, if ΣW_{n+1} be reduced to a single term, $\Sigma\epsilon_n Q_{n+1}$ will at the same time be reduced to a single term containing the same solid harmonic, and ψ_{n+1} and $r^{2n+3}\phi_{-n-2}$ will be the only ψ and ϕ functions that occur.

178. Determination of the unknown harmonics.

We may now suppose that the disturbing potential consists of a single spherical solid harmonic W_{n+1}. Then the ψ and ϕ functions are determined, and likewise the harmonic inequality $\epsilon_n Q_{n+1}$, and we seek to determine the unknown harmonics ΣA_n, ΣB_n, ΣC_n.

From the equation (78) pick out the terms containing spherical solid harmonics of order n, and of order $n+2$. We find two equations

$$-\mu(n-1)A_n\frac{r^n}{a^n}=a_n\frac{\partial Y_{n+1}}{\partial x}+a_n'\epsilon_n\frac{\partial Q_{n+1}}{\partial x}+a_n''\frac{\partial}{\partial x}\left(\frac{r^{2n+3}}{a^{2n+1}}\phi_{-n-2}\right),$$

$$-\mu(n+1)A_{n+2}\frac{r^{n+2}}{a^{n+2}}$$

$$=b_n r^{2n+5}\frac{\partial}{\partial x}\left(\frac{Y_{n+1}}{r^{2n+3}}\right)+b_n'\epsilon_n r^{2n+5}\frac{\partial}{\partial x}\left(\frac{Q_{n+1}}{r^{2n+3}}\right)+b_n''r^{2n+5}\frac{\partial}{\partial x}\left(\frac{\psi_{n+1}}{r^{2n+3}}\right).$$

Simplifying these by means of equations (80) and (81), we may write

$$\left.\begin{aligned}-A_n\frac{r^n}{a^n}&=\frac{1}{(n+1)(2n+1)}\frac{\partial}{\partial x}\left(\frac{r^{2n+3}}{a^{2n+1}}\phi_{-n-2}\right),\\ -A_{n+2}\frac{r^{n+2}}{a^{n+2}}&=\frac{r^{2n+5}}{(n+2)(2n+5)}\frac{\partial}{\partial x}\left(\frac{\psi_{n+1}}{r^{2n+3}}\right)\end{aligned}\right\}\ldots\ldots(84).$$

Since we have already shewn how to express ψ_{n+1} and $\left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2}$ in terms of W_{n+1}, the functions A_n and A_{n+2} are determined, and it is clear that these are the only functions A that occur. The functions B_n, C_n and B_{n+2}, C_{n+2} can be written down by symmetry.

This completes the analytical solution of the problem We shall consider some particular cases.

179. Case where the sphere is not gravitating.

If we annul gravitation in the interior of the sphere the problem is very much simplified. We may replace Y_{n+1} by W_{n+1}, and reject the surface-tractions of art. 172 contributed by the radial strain. We give the results and leave their verification to the reader.

The typical terms of the particular integral for the disturbing forces are given in (73); they are

$$u = -\frac{1}{2(2n+5)} \frac{\rho}{\lambda + 2\mu} \frac{\partial}{\partial x} (r^2 W_{n+1}).$$

The terms contributed to Fr by the particular integrals are given in (74), they are

$$Fr = -\rho \frac{\lambda + \mu(n+2)}{(\lambda + 2\mu)(2n+3)} r^2 \frac{\partial W_{n+1}}{\partial x} + \rho \frac{(2n+5)\lambda + 2(n+2)\mu}{(\lambda + 2\mu)(2n+3)(2n+5)} r^{2n+5} \frac{\partial}{\partial x} \left(\frac{W_{n+1}}{r^{2n+3}} \right).$$

The complementary solutions are the same as those given in art. 176.

The boundary-conditions can be written in the form

$$\Sigma \left[a_n \frac{\partial W_{n+1}}{\partial x} + b_n r^{2n+5} \frac{\partial}{\partial x} \left(\frac{W_{n+1}}{r^{2n+3}} \right) + a_n'' \frac{\partial}{\partial x} \left(\frac{r^{2n+3}}{a^{2n+1}} \phi_{-n-2} \right) + b_n'' r^{2n+5} \frac{\partial}{\partial x} \left(\frac{\psi_{n+1}}{r^{2n+3}} \right) + \mu(n-1) A_n \frac{r^n}{a^n} \right] = 0,$$

when $r = a$; and just as in art. 177 we find

$$\left.\begin{aligned} \psi_{n+1} \left\{ n + 1 + (n+2) \frac{\lambda(n+4) - \mu(n-1)}{\lambda(n+1) + \mu(3n+4)} \right\} \\ = \frac{n+2}{2n+3} \frac{\rho}{\lambda + 2\mu} \frac{(2n+5)\lambda + 2(n+2)\mu}{\mu} W_{n+1} \\ \left(\frac{r}{a} \right)^{2n+3} \phi_{-n-2} = -\frac{n+1}{2n} \frac{2n+1}{2n+3} \frac{\rho}{\lambda + 2\mu} \frac{\lambda + \mu(n+2)}{\mu} W_{n+1} \end{aligned}\right\} \quad \ldots\ldots\ldots(85).$$

The functions A_n, A_{n+2} may then be written down by means of equations (84).

In connexion with this problem we may notice in particular

the case of incompressible material, for which the ratio μ/λ vanishes. We find

$$\psi_{n+1}[n+1+(n+2)(2n+5)E_{n+2}] = \frac{(n+2)(2n+5)}{2n+3}\rho\frac{W_{n+1}}{\mu},$$

$$-\left(\frac{r}{a}\right)^{2n+3}\phi_{-n-2}\frac{2n}{2n+1} = \frac{n+1}{2n+3}\rho\frac{W_{n+1}}{\mu};$$

and equations (84) become

$$\mu A_n\frac{r^n}{a^n} = \frac{\rho a^2}{2n(2n+3)}\frac{\partial W_{n+1}}{\partial x},$$

$$-\mu A_{n+2}\frac{r^{n+2}}{a^{n+2}} = \frac{\rho}{(n+1)+(n+2)(2n+5)E_{n+2}}\frac{r^{2n+5}}{(2n+3)}\frac{\partial}{\partial x}\left(\frac{W_{n+1}}{r^{2n+3}}\right).$$

Comparing these with (43) and (44) we see that the complementary solutions when the displacements are due to a potential W_{n+1} and the surface is free are the same as those produced by purely normal surface-tractions $\rho W_{n+1}\frac{a^{n+1}}{r^{n+1}}$, provided the material of the sphere be incompressible. Now, as in this case the particular integrals (73) are negligible, it follows that purely normal surface-tractions R_n produce the same displacements in an incompressible sphere of radius a as would be produced by bodily forces derivable from a potential $\rho^{-1}R_n(r/a)^n$. This result is otherwise obtained by Mr Chree (*Camb. Phil. Soc. Trans.* XIV. p. 265).

Returning now to the general case we find that the bounding surface, $r=a$, becomes after strain $r=a+\Sigma\epsilon_n Q_{n+1}$, where

$$\epsilon_n Q_{n+1} = a\left[\frac{\psi_{n+1}}{2n+5} - \left(\frac{r}{a}\right)^{2n+3}\frac{\phi_{-n-2}}{2n+1} - \frac{\rho}{\lambda+2\mu}\frac{n+3}{2(2n+5)}W_{n+1}\right].$$

Any other concentric spherical surface $r=r_0$, $(r_0<a)$, becomes

$$r = r_0 + \Sigma\epsilon_n' Q_{n+1}'$$

where

$$\epsilon_n' Q_{n+1}' = \frac{1}{2r_0}\frac{(\lambda+\mu)(n+1)}{\lambda(n+1)+\mu(3n+4)}(a^2-r_0^2)\psi_{n+1}$$

$$+ r_0\left[\frac{\psi_{n+1}}{2n+5} - \left(\frac{r}{a}\right)^{2n+3}\frac{a^2}{r_0^2}\frac{\phi_{-n-2}}{2n+1} - \frac{\rho}{\lambda+2\mu}\frac{n+3}{2(2n+5)}W_{n+1}\right]$$

and we are to give to each spherical harmonic function that occurs its value when $r=r_0$.

Substituting for ψ_{n+1} and ϕ_{-n-2} their values from (85), we find that the height of the harmonic inequality is given by

$$\epsilon_n Q_{n+1} = \frac{a\rho W_{n+1}}{\mu} \frac{n+1}{2n} \frac{(2n+3)\lambda + (2n+2)\mu}{(2n^2+8n+9)\lambda + (2n^2+6n+6)\mu} \quad \ldots(86),$$

and the radial displacement is given by

$$\epsilon_n' Q_{n+1}' = \frac{r_0 \rho W_{n+1}}{\mu} \frac{n+1}{2n} \left[\frac{(2n+3)\lambda + (2n+2)\mu}{(2n^2+8n+9)\lambda + (2n^2+6n+6)\mu} \right.$$
$$\left. + \frac{a^2 - r_0^2}{r_0^2} (n+1) \frac{(n+3)\lambda + (n+2)\mu}{(2n^2+8n+9)\lambda + (2n^2+6n+6)\mu} \right] \ldots(87).$$

The particular case $n = 1$ is interesting. For this case we find

$$\epsilon_1 Q_2 = \frac{5\lambda + 4\mu}{19\lambda + 14\mu} \frac{a\rho W_2}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(88),$$

and

$$\epsilon_1' Q_2' = \left(\frac{8\lambda + 6\mu}{19\lambda + 14\mu} a^2 - \frac{3\lambda + 2\mu}{19\lambda + 14\mu} r_0^2 \right) \frac{\rho W_2}{r_0 \mu} \quad \ldots\ldots\ldots(89).$$

These are equivalent to the expressions otherwise obtained by Sir W. Thomson. Since W_{n+1} is the product of r^{n+1} and a function independent of r, the radial displacement vanishes with r. The result for W_2 can be expressed in the statement: A homogeneous elastic isotropic sphere held strained by balancing attractions from without, is deformed into an harmonic spheroid, of the same type as the potential of the disturbing forces, and all the concentric spherical surfaces are deformed into harmonic spheroids of the same type. These surfaces are not similar, but the ellipticities of all the principal sections increase from the outermost to the centre, the ratio of the extreme values being $(5\lambda + 4\mu)/(8\lambda + 6\mu)$[1].

180. Gravitating nearly spherical mass.

Another simple case is that of a nearly spherical mass held strained by its own gravitation. Suppose the strained form is

$$r = a + \epsilon_n Q_{n+1},$$

where ϵ_n is small and Q_{n+1} is a spherical solid harmonic, of order $n+1$. The potential of the bodily forces is

$$-\tfrac{1}{2} g \frac{r^2}{a} + \frac{3g}{2n+3} \epsilon_n Q_{n+1}$$

so that we have $3g\epsilon_n Q_{n+1}/(2n+3)$ instead of Y_{n+1}.

[1] Thomson and Tait, *Natural Philosophy*, Part II. art. 835.

The terms contributed to Fr by the particular integral for this bodily force are given by

$$Fr = -\rho \frac{\lambda + (n+2)\mu}{(\lambda+2\mu)(2n+3)} \frac{3g\epsilon_n}{2n+3} a^2 \frac{\partial Q_{n+1}}{\partial x} + \rho \frac{(2n+5)\lambda + (2n+4)\mu}{(\lambda+2\mu)(2n+3)(2n+5)} \frac{3g\epsilon_n r^{2n+5}}{2n+3} \frac{\partial}{\partial x}\left(\frac{Q_{n+1}}{r^{2n+3}}\right).$$

The terms contributed to Fr by taking the stress produced by the radial force of gravitation at the deformed surface are given by

$$Fr = \frac{g\rho}{5(\lambda+2\mu)} \frac{\epsilon_n}{2n+3} \left[\{5\lambda + 2(n+5)\mu\} a^2 \frac{\partial Q_{n+1}}{\partial x} - \{5\lambda - (2n-4)\mu\} r^{2n+5} \frac{\partial}{\partial x}\left(\frac{Q_{n+1}}{r^{2n+3}}\right) \right].$$

The terms contributed to Fr by the complementary functions are given by (77).

Thus collecting the terms the boundary-condition can be written

$$\alpha_n \frac{\partial Q_{n+1}}{\partial x} + \beta_n r^{2n+5} \frac{\partial}{\partial x}\left(\frac{Q_{n+1}}{r^{2n+3}}\right) + \beta_n' r^{2n+5} \frac{\partial}{\partial x}\left(\frac{\psi_{n+1}}{r^{2n+3}}\right) + \alpha_n'' \frac{\partial}{\partial x}(r^{2n+3}\phi_{-n-2}) + \mu\Sigma(n-1) A_n \frac{r^m}{a^n} = 0 \quad \ldots\ldots\ldots (90),$$

where

$$\left.\begin{aligned} \alpha_n &= \frac{g\rho\epsilon_n a^2}{(2n+3)(\lambda+2\mu)} \left\{ \frac{5\lambda + 2(n+5)\mu}{5} - 3\frac{\lambda + (n+2)\mu}{2n+3} \right\}, \\ \beta_n &= \frac{g\rho\epsilon_n}{(2n+3)(\lambda+2\mu)} \left\{ 3\frac{(2n+5)\lambda + (2n+4)\mu}{(2n+3)(2n+5)} - \frac{5\lambda - (2n-4)\mu}{5} \right\}, \\ \alpha_n'' &= -\frac{\mu}{(2n+1)a^{2n+1}}, \\ \beta_n' &= -\frac{\mu}{2n+5} \frac{\lambda(n+4) - \mu(n-1)}{\lambda(n+1) + \mu(3n+4)} \end{aligned}\right\} \quad \ldots\ldots\ldots (91).$$

The three equations such as (90) give

$$\left.\begin{aligned} &\left\{\frac{n+1}{n+2} + \frac{\lambda(n+4) - \mu(n-1)}{\lambda(n+1) + \mu(3n+4)}\right\} \mu\psi_{n+1} \\ &= \frac{2n+5}{2n+3} \frac{g\rho\epsilon_n Q_{n+1}}{\lambda+2\mu} \left\{ 3\frac{(2n+5)\lambda + (2n+4)\mu}{(2n+3)(2n+5)} - \frac{5\lambda - (2n-4)\mu}{5} \right\}, \\ &\frac{2n}{2n+1} \mu \left(\frac{r}{a}\right)^{2n+3} \phi_{-n-2} \\ &= \frac{(n+1)}{2n+3} \frac{g\rho\epsilon_n Q_{n+1}}{\lambda+2\mu} \left\{ \frac{5\lambda + 2(n+5)\mu}{5} - 3\frac{\lambda + (n+2)\mu}{2n+3} \right\} \end{aligned}\right\} \quad \ldots\ldots\ldots (92).$$

Thus ψ_{n+1}, and ϕ_{-n-2}, are determined in terms of $\epsilon_n Q_{n+1}$, and the A's are given by the equations

$$\left.\begin{aligned} -\mu(n-1)A_n\frac{r^n}{a^n} &= \alpha_n\frac{\partial Q_{n+1}}{\partial x}+\alpha_n''\frac{\partial}{\partial x}(r^{2n+3}\phi_{-n-2}) \\ -\mu(n+1)A_{n+2}\frac{r^{n+2}}{a^{n+2}} &= r^{2n+5}\left\{\beta_n\frac{\partial}{\partial x}\left(\frac{Q_{n+1}}{r^{2n+3}}\right)+\beta_n'\frac{\partial}{\partial x}\left(\frac{\psi_{n+1}}{r^{2n+3}}\right)\right\} \end{aligned}\right\} \quad (93).$$

The displacement u is given by the equation

$$\begin{aligned} u = Ax + Hr^2x + A_n\frac{r^n}{a^n} + A_{n+2}\frac{r^{n+2}}{a^{n+2}} \\ + (a^2-r^2)\tfrac{1}{2}\frac{\lambda+\mu}{\lambda(n+1)+\mu(3n+4)}\frac{\partial\psi_{n+1}}{\partial x} \\ - \frac{\rho}{(\lambda+2\mu)}\frac{1}{2(2n+5)}\frac{3g\epsilon_n}{2n+3}\frac{\partial}{\partial x}(r^2Q_{n+1}) \quad \ldots\ldots(94). \end{aligned}$$

This, and the similar expressions which can be written down from symmetry, constitute the complete solution of the problem.

Prof. G. H. Darwin[1] has used the solution of this problem to find an expression for the stresses, produced in the interior of the earth by the weight of continents, and thence to obtain an estimate of the strength of the materials of the earth. Mr Chree[2] has shewn that if the material be regarded as incompressible, so that μ/λ is very small, then the tendency to rupture as measured by the difference of the greatest and least principal stresses (Prof. Darwin's measure) depends on the harmonic inequality $\epsilon_n Q_{n+1}$, *i.e.* the question can be discussed by the aid of the above or a similar analysis; if μ/λ be not very small, the maximum stress-difference depends on the radial strain. The same writer has also shewn that, if μ/λ be very large or very small, the tendency to rupture, as measured by the greatest principal extension, would again depend on the harmonic inequality, but unless μ/λ be very large or very small it depends on the radial strain. When μ and λ are comparable we have seen already (art. 127) that the materials of the earth, regarded as homogeneous and isotropic, would have to be very much stronger than any known material in order to resist the tendency to rupture near the surface, arising from gravitation. Prof. Darwin's conclusion as to the great strength of the materials of the earth appears to require some modification, depending on

[1] *Phil. Trans. R. S.* 1882, pp. 187 sq.

[2] *Camb. Phil. Trans.* XIV. 1887, pp. 278 sq. and *Phil. Mag.* XXXII. 1891.

the internal heterogeneity. An account of his results is given in Thomson and Tait's *Nat. Phil.* part II. art. 832′.

181. Disturbing Potential a spherical harmonic of the second order.

The cases of the general problem of art. 170 of greatest interest are those in which the disturbing potential is a spherical solid harmonic of order 2. These include the theory of the equilibrium figure of the rotating sphere, and the theory of the bodily tides in an elastic solid earth.

Suppose then that $n = 1$ and seek to determine $\epsilon_1 Q_2$ the height of the harmonic inequality. We have to use the equations obtained from (82) and (83), viz.:

$$\left.\begin{aligned} \mu\psi_2 [2 + 21E_3] &= 21 [b_1'\epsilon_1 Q_2 + b_1 (W_2 + \tfrac{3}{5} g\epsilon_1 Q_2)], \\ \mu \left(\frac{r}{a}\right)^5 \phi_{-3} &= \frac{3}{a^2} [a_1'\epsilon_1 Q_2 + a_1 (W_2 + \tfrac{3}{5} g\epsilon_1 Q_2)], \\ \frac{\psi_2}{7} - \left(\frac{r}{a}\right)^5 \frac{\phi_{-3}}{3} - \tfrac{2}{7} \frac{\rho}{\mu + 2\mu} (W_2 + \tfrac{3}{5} g\epsilon_1 Q_2) &= (1 - A - 3Ha^2) \frac{\epsilon_1 Q_2}{a} \end{aligned}\right\} \quad (95).$$

The constants H, A, E_3, a_1, a_1', b_1, b_1' are given by the equations

$$\left.\begin{aligned} H &= \tfrac{1}{10} \frac{g\rho}{(\lambda + 2\mu)\, a}, \\ A &= -\frac{5\lambda + 6\mu}{3\lambda + 2\mu} \tfrac{1}{10} \frac{g\rho a}{\lambda + 2\mu}, \\ E_3 &= \tfrac{1}{7} \frac{5\lambda}{2\lambda + 7\mu}, \\ a_1 &= -\tfrac{1}{5} \frac{\lambda + 3\mu}{\lambda + 2\mu} \rho a^2, \\ a_1' &= \tfrac{1}{25} \frac{5\lambda + 12\mu}{\lambda + 2\mu} g\rho a^2, \\ b_1 &= \tfrac{1}{35} \frac{7\lambda + 6\mu}{\lambda + 2\mu} \rho, \\ b_1' &= -\tfrac{1}{25} \frac{5\lambda + 2\mu}{\lambda + 2\mu} g\rho \end{aligned}\right\} \quad \text{.....(96),}$$

where the terms in ω have been rejected from H and A as on p. 289.

We shall consider the particular cases where $\mu/\lambda = 0$ and $\mu/\lambda = 1$.

182. Incompressible material.

When the material of the sphere is incompressible we have λ infinite and the constants become

$$A=0,\ H=0,\ E_3=\tfrac{5}{14},$$

$$a_1=-\tfrac{1}{5}\rho a^2,\ a_1'=\tfrac{1}{5}g\rho a^2,\ b_1=\tfrac{1}{5}\rho,\ b_1'=-\tfrac{1}{5}g\rho.$$

Hence the first two of equations (95) become

$$\tfrac{19}{2}\mu\psi_2=\tfrac{21}{5}\rho\,(W_2-\tfrac{2}{5}g\epsilon_1Q_2),$$

$$\mu\left(\frac{r}{a}\right)^5\phi_{-3}=-\tfrac{3}{5}\rho\,(W_2-\tfrac{2}{5}g\epsilon_1Q_2);$$

and the third of equations (95) then gives

$$\epsilon_1Q_2=\frac{a\rho W_2}{\dfrac{2g\rho a}{5}+\dfrac{19\mu}{5}}.$$

We shall write this result

$$\epsilon_1Q_2=\frac{15\vartheta}{6\vartheta+19}\,\frac{W_2}{g} \qquad \text{..... (97),}$$

where

$$\vartheta=\tfrac{1}{3}g\rho a/\mu,$$

so that $(3\vartheta)^{-\frac{1}{2}}$ is the ratio of the velocity of waves of distortion in the material to that due to falling through half the radius of the sphere under gravity kept constant and equal to that at the surface.

183. Material fulfilling Poisson's condition.

When the material fulfils Poisson's condition ($\lambda=\mu$) the constants become

$$A=-\tfrac{11}{50}\vartheta,\ H=\tfrac{1}{10}\frac{\vartheta}{a^2},\ E_3=\tfrac{5}{63},$$

$$a_1=-\tfrac{4}{5}\frac{\mu\vartheta a}{g},\ a_1'=\tfrac{17}{25}\mu\vartheta a,\ b_1=\tfrac{13}{35}\frac{\mu\vartheta}{ag},\ b_1'=-\tfrac{7}{25}\frac{\mu\vartheta}{a},$$

where ϑ is the number defined in the last article.

Also the first two of equations (95) become

$$\tfrac{11}{3}\psi_2=21\,\frac{\vartheta}{a}\left[\tfrac{13}{35}\left(\frac{W_2}{g}+\tfrac{3}{5}\epsilon_1Q_2\right)-\tfrac{7}{25}\epsilon_1Q_2\right],$$

$$\left(\frac{r}{a}\right)^5\phi_{-3}=3\,\frac{\vartheta}{a}\left[-\tfrac{4}{5}\left(\frac{W_2}{g}+\tfrac{3}{5}\epsilon_1Q_2\right)+\tfrac{17}{25}\epsilon_1Q_2\right].$$

Hence

$$\frac{\psi_2}{7} - \left(\frac{r}{a}\right) \frac{\phi_{-3}}{3} = \tfrac{85}{77} \frac{\mathfrak{S}}{ag} W_2 - \tfrac{19}{77} \frac{\mathfrak{S}}{a} \epsilon_1 Q_2,$$

so that the third of equations (95) becomes

$$\tfrac{85}{77} \frac{\mathfrak{S}}{ag} W_2 - \tfrac{19}{77} \frac{\mathfrak{S}}{a} \epsilon_1 Q_2 - \tfrac{2}{7} \frac{\mathfrak{S}}{a} \left(\frac{W_2}{g} + \tfrac{3}{5} \epsilon_1 Q_2\right) = \left(1 - \frac{2\mathfrak{S}}{25}\right) \frac{\epsilon_1 Q_2}{a}.$$

From this equation we obtain the height of the harmonic inequality

$$\epsilon_1 Q_2 = \frac{225\mathfrak{S}}{275 + 93\mathfrak{S}} \frac{W_2}{g} \quad \text{.....(98).}$$

The result (97) of the last article is that when the material is incompressible the height of the harmonic inequality is

$$\epsilon_1 Q_2 = \frac{225\mathfrak{S}}{285 + 90\mathfrak{S}} \frac{W_2}{g}.$$

The difference between the result which holds when the material is incompressible and that which holds when the material fulfils Poisson's condition is a very small fraction of either for any the same value of $\mathfrak{S}$, so that in case it is uncertain which hypothesis is the best to make no very great error can arise in our estimate of the harmonic inequality if we assume the material to be incompressible.

In the applications that we shall make to problems relating to the earth, considered as an elastic solid globe, we shall have to assume the material incompressible to avoid the difficulties explained in art. 127.

184. Rotating Sphere.

Consider the problem of a solid globe of incompressible elastic material rotating with angular velocity ω.

We have $$W_2 = -\tfrac{1}{3} \omega^2 r^2 P_2,$$

where P_2 is Legendre's second coefficient

$$\tfrac{3}{2} \cos^2 \theta - \tfrac{1}{2};$$

and $\epsilon_1 Q_2$ is given by

$$\epsilon_1 Q_2 = -\tfrac{1}{3} \frac{\omega^2}{g} r^2 P_2 \frac{15\mathfrak{S}}{6\mathfrak{S} + 19}.$$

The equation of the surface is

$$r = a\left[1 - \tfrac{2}{5}P_2 \frac{\omega^2 a}{g} \frac{\frac{5}{4}}{1 + \frac{19}{2}\frac{\mu}{g\rho a}}\right] \quad \ldots\ldots\ldots\ldots\ldots(99),$$

so that the ellipticity of the surface is

$$\tfrac{5}{4}\frac{\omega^2 a}{g} \frac{1}{1 + \frac{19}{2}\frac{\mu}{g\rho a}} \quad \ldots\ldots\ldots\ldots\ldots(100).$$

For a liquid sphere the ellipticity would be $\frac{5}{4}\omega^2 a/g$.

If the globe be of the same mass and diameter as the Earth, and of the rigidity of steel or iron, we have

$a = 640 \times 10^6$ centimetres,

$\rho = 5{\cdot}5$,

$\mu = 780 \times 10^6$ grammes' weight per sq. centimetre[1].

The ellipticity of the surface due to the rotation is diminished by the rigidity in the ratio

$$1 : 1 + \tfrac{19}{2}\frac{780}{11\frac{1}{2}\,640}$$

or nearly 1 : 3.

If the rigidity were that of glass,

$\mu = 244 \times 10^6$ grammes' weight per square centimetre;

and the ratio is $$1 : 1 + \tfrac{19}{2}\frac{244}{11\frac{1}{2}\,640}$$

or nearly 3 : 5.

The same numbers apply generally to a globe of the same mass and diameter of the Earth whatever may be the forces whose potential is W_2, and we have

$\mathfrak{Y} = \frac{3}{2}$ nearly for a rigidity equal to that of steel,

$\mathfrak{Y} = 5$ nearly for a rigidity equal to that of glass.

185. Tide-generating Forces.

The attraction of the Moon or any distant body at any point within the Earth's surface can be regarded as compounded of a radial force between the centres of the two bodies and forces

[1] This is the value for wrought iron given in the table, p. 77.

which vary from point to point. The first produces a motion of the centre of gravity of the Earth, and the remaining forces are a system which applied to a rigid body would produce equilibrium. Applied to the Earth, they produce small relative motions of its parts, which, by analogy to the corresponding motions of the Sea relative to the Earth, may be called *tides*. Now the tide-generating forces are derivable from a potential expressible in spherical harmonic series, and the most important terms are those of the second order. (See Thomson and Tait, *Nat. Phil.*, Part II., arts. 804 sq.) The expression for W_2 the tide-generating potential referred to the line joining the centres of the Earth and Moon as axis of the harmonic is

$$W_2 = \frac{M\gamma a^2}{D^3}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

correct to terms of the second order, where M is the Moon's mass, a the Earth's radius, D the Moon's distance, and γ the constant of gravitation. When this expression is referred to axes fixed in the Earth, it still consists solely of spherical harmonics of the second order, but the coefficients of these are periodic functions of the time. The principal terms are diurnal and semi-diurnal, depending on the rotation of the Earth, fortnightly and monthly depending on the motion of the Moon in its orbit, and nineteen-yearly depending on periodic changes in this orbit characterised by the revolution of the nodes in the ecliptic.

The Sun produces tides as well as the Moon, and the tide-generating forces have periodic variations of semi-annual and annual periods, depending on the motion of the Earth in its orbit.

The complete expression for the tide-generating potential therefore consists of a sum of spherical harmonics of the second order, and these have coefficients among which there is one with a semi-diurnal period, one with a diurnal period, and so on. Each of these terms would produce in a liquid globe, or in a mass of liquid resting on a rigid spherical nucleus, a deformation of the surface into an harmonic spheroid of the second order (called a "tide"), and the ellipticity of the spheroid would be proportional to the corresponding term of the tide-generating potential, and would be a periodic function of the time with a period coincident with that of the term. We therefore speak of diurnal and semi-

diurnal tides, fortnightly and monthly tides, annual and semi-annual tides, and of a nineteen-yearly tide.

186. Elastic Tides in Solid Earth.

Exactly the same kind of deformation would be experienced by an elastic solid globe, and we have seen how the elevation $\epsilon_1 Q_2$ of the surface can be expressed in terms of the tide-generating potential W_2.

If the globe be homogeneous and incompressible, of radius a, density ρ, and rigidity μ, the ellipticity of the surface is

$$\frac{1}{1+\frac{19}{2}\frac{\mu}{g\rho a}}$$

of that in a liquid globe of the same size and density.

Sir W. Thomson calls attention to the smallness of the part played by rigidity, as compared with gravity, in resisting the deforming influence. We can see by using the results of art. 184 that the ratio of the ellipticities for a liquid globe and one as rigid as steel and incompressible is only about 3, and it is only about $\frac{5}{3}$ when the rigidity is that of glass.

The height of the tide, measured by rise and fall of sea relative to land, is reduced by the elastic yielding of the nucleus to the fraction

$$\frac{\frac{19}{2}\frac{\mu}{g\rho a}}{1+\frac{19}{2}\frac{\mu}{g\rho a}}$$

of the true equilibrium height, the material being incompressible.

This ratio is about $\frac{2}{3}$ when μ is the same as that for steel, and about $\frac{2}{5}$ when μ is the same as that for glass.

187. Tidal Effective Rigidity.

Sir W. Thomson has applied the calculation, in the case of incompressible material, to test the geological hypothesis of the internal fluidity of the Earth. The problem may be stated thus:—

Supposing that, for purposes of discussion, the Earth is re-

placed by a homogeneous incompressible elastic solid globe of the same mass and diameter, what degree of rigidity must be ascribed to the solid, in order that ocean-tides upon it may be of the same height as those on the Earth?

If this question were answered, the rigidity found would be that which Sir W. Thomson calls the *tidal effective rigidity* of the Earth.

There are many difficulties in the way of a complete answer to this question. In the first place we have here investigated only the *equilibrium* of the sphere under bodily forces, and therefore the tide considered must be one that follows very nearly the equilibrium law. The diurnal and semi-diurnal tides may therefore be dismissed. We shall see hereafter that the longest period of free vibration of the sphere, (supposed as rigid as steel,) in which its surface would be deformed according to a spherical harmonic of the second order, is 1 hr. 6 min. and thus an equilibrium theory would apply to fortnightly tides in the elastic solid globe. It has however been pointed out by Prof. G. H. Darwin[1] that it is very doubtful whether such a theory applies to the fortnightly ocean-tides. That it may do so requires a very great frictional resistance at the ocean-bed, much greater than is considered probable. There remain the tides of long period, the annual and semi-annual tides and the minute nineteen-yearly tide. The former are difficult to estimate on account of annual fluctuations of ocean-level, due to the melting of ice in the polar regions; the latter is probably too small to be observed. So far the difficulties of the tidal theory.

Supposing these difficulties could be surmounted, and the *tidal effective rigidity* determined, we should still have to consider what light the determination throws upon the question of internal fluidity. The Earth is not a *homogeneous elastic solid* globe, its material is heterogeneous, and it is conceivable that a much smaller degree of rigidity of the materials in a heterogeneous globe might suffice to produce considerable resistance to deformation of the surface than would be required if the material were homogeneous. This matter has not been considered mathematically; but until it is settled it remains open to question whether

[1] 'On the Dynamical Theory of the tides of long period'. *Proc. R. S.* 1886.

the *tidal effective rigidity* defined above throws any light on the hypothesis of the internal fluidity of the Earth. We may observe that heterogeneity has a marked effect on the ellipticity of the surface of rotating incompressible fluid. For a homogeneous liquid globe of the same size and mass as the Earth, rotating once in 24 hrs., the ellipticity of the surface is about $\frac{1}{230}$, while there is no difficulty in inventing a law of density which shall make it equal to $\frac{1}{297}$, the observed value in the case of the Earth. This shews that the part of the resistance to deformation arising from gravity can be considerably increased by supposing the material heterogeneous, but it is not at all clear *à priori* how the resistance depending on the rigidity would be influenced by heterogeneity.

188. Rigidity of the Earth.

If the Earth be regarded as homogeneous, and incompressible, and of rigidity equal to that of steel, the height of the ocean-tides is reduced by the elastic yielding to $\frac{2}{3}$ of the true equilibrium height. If the rigidity be that of glass, the fraction is $\frac{2}{5}$.

From certain observations made in the Indian Ocean, Prof. G. H. Darwin concluded[1] that the observable fortnightly tide is really not much less than $\frac{2}{3}$, and certainly much greater than $\frac{2}{5}$ of the true equilibrium height; and Sir W. Thomson argued thence that the tidal effective rigidity of the Earth must be much greater than the rigidity of glass, and very nearly as great as that of steel. He has on this and other independent grounds held that the geological hypothesis of internal fluidity is disproved. The difficulties we have pointed out in the last article appear to lead to the conclusion that, in the present state of knowledge, tidal phenomena do not yield any result which we can apply in a satisfactory manner to test this hypothesis. Prof. G. H. Darwin in his most recent work[2] upon the subject is of the same opinion, viz., that tidal theory is not decisive either for or against the hypothesis.

[1] See Thomson and Tait, *Nat. Phil.* Part II., art. 848.

[2] *Proc. R. S.* 1886.

CHAPTER XI.

VIBRATIONS OF A SPHERE[1].

189. THE problem of determining the normal modes and periods of vibration of an isotropic elastic solid sphere or spherical shell whose surface is free was first completely solved by Prof. Lamb. It is a most interesting example of the general theory of the free vibrations of solids explained in arts. 79 and 80.

We shall consider, in the first place, the theory of the free vibrations of a solid sphere or spherical shell, and afterwards the problem of forced vibrations in a solid sphere produced by forces derivable from a potential expressible in spherical harmonic series.

190. Differential equations of Free Vibration.

We have to find solutions of the equations of displacement

$$\left.\begin{aligned}(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u&=\rho\frac{\partial^2 u}{\partial t^2},\\ (\lambda+\mu)\frac{\partial\Delta}{\partial y}+\mu\nabla^2 v&=\rho\frac{\partial^2 v}{\partial t^2},\\ (\lambda+\mu)\frac{\partial\Delta}{\partial z}+\mu\nabla^2 w&=\rho\frac{\partial^2 w}{\partial t^2}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

[1] The following among other authorities may be consulted:

Jaerisch 'Ueber die elastischen Schwingungen einer isotropen Kugel'. Crelle's *Journal*, LXXXVIII. 1880.

Lamb 'On the Vibrations of an elastic sphere'. *Proc. Lond. Math. Soc.* XIII. 1882, and 'On the Vibrations of a spherical shell'. *Proc. Lond. Math. Soc.* XIV. 1883.

Love 'The free and forced Vibrations of an elastic spherical shell...'. *Proc. Lond. Math. Soc.* XIX. 1888.

Chree 'On the equations of an isotropic elastic solid in cylindrical and polar coordinates'. *Camb. Phil. Soc. Trans.* XIV. 1887.

Rayleigh 'On Waves propagated along the plane surface of an elastic solid'. *Proc. Lond. Math. Soc.* XVII. 1886.

which are simple harmonic functions of the time, are finite, continuous, and one-valued within the boundary, and satisfy the condition that the bounding surface is free from stress.

Suppose the solid performing free vibrations whose period is $2\pi/p$; then for $\frac{\partial^2 u}{\partial t^2}\ldots$ we may substitute $-p^2u\ldots$, and thus the equations (1) become of the type

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u+\rho p^2 u=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Differentiating these with respect to x, y, z, adding, and writing

$$h^2=p^2\rho/(\lambda+2\mu), \quad \kappa^2=p^2\rho/\mu \quad \ldots\ldots\ldots\ldots\ldots(4),$$

we have

$$(\nabla^2+h^2)\,\Delta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and the equations can be written

$$\left.\begin{aligned}(\nabla^2+\kappa^2)\,u&=(1-\kappa^2/h^2)\frac{\partial\Delta}{\partial x},\\ (\nabla^2+\kappa^2)\,v&=(1-\kappa^2/h^2)\frac{\partial\Delta}{\partial y},\\ (\nabla^2+\kappa^2)\,w&=(1-\kappa^2/h^2)\frac{\partial\Delta}{\partial z}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(6).$$

Equations (6) can be satisfied by putting

$$u=-\frac{1}{h^2}\frac{\partial\Delta}{\partial x}, \quad v=-\frac{1}{h^2}\frac{\partial\Delta}{\partial y}, \quad w=-\frac{1}{h^2}\frac{\partial\Delta}{\partial z} \quad \ldots\ldots\ldots(7),$$

where Δ satisfies (5), and these satisfy (2).

Hence the complete solutions of the equations of vibration consist of the sums of these solutions and the general solutions of the equations

$$\left.\begin{aligned}(\nabla^2+\kappa^2)\,u=0, \quad (\nabla^2+\kappa^2)\,v=0, \quad (\nabla^2+\kappa^2)\,w=0,\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\end{aligned}\right\}\quad\ldots\ldots(8).$$

191. Description of method.

Before proceeding we point out the kind of results to be obtained. According to the theory explained in art. 79, there will be an indefinite number of normal modes of oscillation, and the oscillations of any normal mode can be executed independently.

If the system be oscillating in a normal mode then at any instant the displacements can be expressed in the form

$$u = u'A\cos(pt+\epsilon), \quad v = v'A\cos(pt+\epsilon), \quad w = w'A\cos(pt+\epsilon),$$

where $2\pi/p$ is the period, A is a small arbitrary constant, and u', v', w' are functions of x, y, z. These functions are called *normal functions*, and the determination of the vibrations of any elastic system is effected when the normal functions are known and the frequency-equations have been formed and solved. In what follows we shall first determine the forms of the normal functions; and no confusion ought to arise if we denote them by u, v, w, instead of u', v', w', and write Δ for $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z$, where u, v, w are simply normal functions. In strictness each term of the cubical dilatation also contains a factor of the form $A\cos(pt+\epsilon)$.

Among the vibrations of a sphere we shall find that for some modes there are spherical surfaces at which the displacement vanishes, just as in the vibrations of a string there may be one or more nodal points. Such surfaces will be called *nodal* surfaces, and their number and position are determined by the type of vibration and the frequency, and, conversely, if the number and position of these surfaces be given the type and the frequency are determinate. We shall find also other modes for which there are no surfaces at which the displacement vanishes, but there will then be surfaces at which the radial displacement vanishes, and we shall term such surfaces *quasi-nodal*. The number of the quasi-nodal surfaces for a particular class of vibrations does not in general determine the frequency or the type.

We proceed now to the consideration of the vibrations of an isotropic elastic sphere.

192. Determination of the Dilatation.

We have to find a solution of the equation $(\nabla^2 + h^2)\Delta = 0$ in a form adapted to satisfy boundary-conditions at the surface of a sphere. We therefore suppose Δ, at the surface of the sphere, expressible in spherical surface-harmonics, and we treat the typical term $\Delta = R_n S_n$, where S_n is a spherical surface-harmonic, and R_n is a function of r, defined by the equation

$$\frac{\partial^2}{\partial(hr)^2}(rR_n) + rR_n - \frac{n(n+1)}{(hr)^2}(rR_n) = 0 \ldots\ldots\ldots (9).$$

This is the case of Riccati's equation which is integrable in

terms of circular functions, and the solution which remains finite in space containing the origin is

$$R_n = r^n \left(\frac{1}{r}\frac{d}{dr}\right)^n \left(\frac{\sin hr}{r}\right) \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

This function can be expanded in a convergent series of powers of r, beginning with r^n, and, if we take such a multiplier as will make the first coefficient unity, and write $r^n S_n = \omega_n$, we shall have as the general form of Δ

$$\Delta = \sum_{n=0}^{n=\infty} \omega_n \psi_n (hr) \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where ω_n is a spherical solid harmonic of order n, and

$$\psi_n (x) = (-)^n 1.3.5\ldots(2n+1) \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \ldots (12).$$

We add a few properties of the functions $\psi_n (x)$ which admit of ready verification:

The equations connecting consecutive ψ's are

$$x\frac{d}{dx}\psi_{n-1}(x) = -\frac{x^2}{2n+1}\psi_n(x) = (2n-1)\{\psi_{n-2}(x) - \psi_{n-1}(x)\} \ldots (13).$$

The differential equation is

$$\frac{d^2\psi_n(x)}{dx^2} + \frac{2(n+1)}{x}\frac{d\psi_n(x)}{dx} + \psi_n(x) = 0 \ldots\ldots\ldots(14).$$

The series for $\psi_n (x)$ is

$$\psi_n(x) = 1 - \frac{x^2}{2.2n+3} + \frac{x^4}{2.4.(2n+3)(2n+5)} - \ldots (15);$$

and thus $\psi_n (x) = \frac{1}{2}\sqrt{(2\pi)}\, 1.3\ldots(2n+1)\, x^{-(n+\frac{1}{2})} J_{n+\frac{1}{2}}(x) \ldots (16)$, where $J_{n+\frac{1}{2}}(x)$ is the Bessel's function of order $n+\frac{1}{2}$.

The function $\psi_n (z)$ of the complex variable z is a uniform function in all parts of the plane of z which exclude the point at infinity. This point is an essential critical point of the function.

193. Determination of the Displacements.

The forms of u, v, w, which satisfy (8), can be written down in the same way; thus

$$u = \Sigma U_n \psi_n (\kappa r),\ v = \Sigma V_n \psi_n (\kappa r),\ w = \Sigma W_n \psi_n (\kappa r) \ldots\ldots(17),$$

where U_n, V_n, W_n are spherical solid harmonics of order n, and these have to be arranged to satisfy the condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \ldots\ldots\ldots\ldots\ldots(18).$$

Now this condition is

$$\Sigma\left[\psi_n(\kappa r)\left(\frac{\partial U_n}{\partial x}+\frac{\partial V_n}{\partial y}+\frac{\partial W_n}{\partial z}\right)+\frac{\partial \psi_n(\kappa r)}{\partial r}\left(\frac{xU_n+yV_n+zW_n}{r}\right)\right] = 0 \ldots\ldots\ldots (19).$$

If χ_n be a spherical solid harmonic of order n, then the forms

$$U_n = y\frac{\partial \chi_n}{\partial z}-z\frac{\partial \chi_n}{\partial y},\quad V_n = z\frac{\partial \chi_n}{\partial x}-x\frac{\partial \chi_n}{\partial z},\quad W_n = x\frac{\partial \chi_n}{\partial y}-y\frac{\partial \chi_n}{\partial x}$$

satisfy the equations

$$xU_n+yV_n+zW_n = 0, \text{ and } \partial U_n/\partial x+\partial V_n/\partial y+\partial W_n/\partial z = 0,$$

and U_n, V_n, W_n are spherical solid harmonics of order n.

Again if U_n be $\dfrac{\partial \phi_{n+1}}{\partial x}-\alpha_n r^{2n+1}\dfrac{\partial}{\partial x}\left(\dfrac{\phi_{n-1}}{r^{2n-1}}\right)$, and V_n and W_n be similar expressions with y and z respectively for x, these will be spherical solid harmonics of order n provided ϕ_{n+1} be one of order $n+1$, and we shall have

$$\frac{\partial U_n}{\partial x}+\frac{\partial V_n}{\partial y}+\frac{\partial W_n}{\partial z} = n(2n+1)\alpha_n\phi_{n-1},$$

and $$xU_n+yV_n+zW_n = (n+1)\phi_{n+1}+\alpha_n r^2 n\phi_{n-1}.$$

Thus the terms contributed to (19) by such functions U_n, V_n, W_n, will contain ϕ_{n+1} multiplied by

$$\frac{n+1}{r}\frac{\partial \psi_n}{\partial r}+\alpha_{n+2}(n+2)r\frac{\partial \psi_{n+2}}{\partial r}+(n+2)(2n+5)\alpha_{n+2}\psi_{n+2},$$

where ψ_n is written for $\psi_n(\kappa r)$.

By using (13) the multiplier becomes

$$\left[-(n+1)\frac{\kappa^2}{2n+3}+(n+2)(2n+5)\alpha_{n+2}\right]\psi_{n+1}.$$

This vanishes identically if we take

$$\alpha_{n+2} = \frac{n+1}{n+2}\frac{\kappa^2}{(2n+3)(2n+5)}.$$

Thus we have found solutions of (8) in the form

$$u = \Sigma\left[\psi_n(\kappa r)\left(\frac{\partial \phi_{n+1}}{\partial x}+y\frac{\partial \chi_n}{\partial z}-z\frac{\partial \chi_n}{\partial y}\right)\right.$$
$$\left.-\frac{n+1}{n+2}\frac{\kappa^2 r^{2n+5}}{(2n+3)(2n+5)}\psi_{n+2}(\kappa r)\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right)\right] \ldots\ldots (20),$$

where v and w are to be derived from this by cyclical interchanges of the letters x, y, z, and χ_n and ϕ_{n+1} are spherical solid harmonics of orders indicated by the suffixes. This solution contains two

unknown spherical harmonic functions, and therefore constitutes the general solution of the form required.

The complete expressions for the normal functions will be found by adding together the left-hand sides of the equations such as (20), and the particular solutions given by (7) and (11). They depend upon three sets of unknown spherical solid harmonics, ω, χ, ϕ, and we shall shew how to determine the ratios of these from the boundary-conditions.

For convenience of reference we state here the results so far obtained. The cubical dilatation is

$$\Sigma A \omega_n \psi_n (hr) \cos (pt + \epsilon).$$

The displacement u is

$$\Sigma A \cos (pt + \epsilon) \left[-\frac{1}{h^2} \frac{\partial}{\partial x} \{\omega_n \psi_n (hr) + \psi_n (\kappa r) \left(y \frac{\partial \chi_n}{\partial z} - z \frac{\partial \chi_n}{\partial y} \right) \right.$$
$$\left. + \psi_{n-1} (\kappa r) \frac{\partial \phi_n}{\partial x} - \psi_{n+1} (\kappa r) \frac{n}{n+1} \frac{\kappa^2 r^{2n+3}}{(2n+1)(2n+3)} \frac{\partial}{\partial x} \left(\frac{\phi_n}{r^{2n+1}} \right) \right],$$

and the displacements v and w are to be derived by cyclical interchanges of the letters x, y, z. The summations extend to all integral values of n, and to all values of p given by the frequency-equations with the corresponding values of h and κ.

It is also convenient to state that the product of the radius-vector and the radial displacement is given by the equation

$$\zeta = ux + vy + wz = -\frac{1}{h^2} \Sigma A \cos (pt + \epsilon) [\{n \psi_n (hr) + hr \psi_n' (hr)\} \omega_n]$$
$$+ \Sigma A \cos (pt + \epsilon) [n \psi_n (\kappa r) \phi_n].$$

194. Surface-tractions depending on Dilatation.

We saw in art. 167 that, if F, G, H be the surface-tractions on a sphere of radius r,

$$Fr = \lambda x \Delta + \mu \frac{\partial}{\partial x} (ux + vy + wz) + \mu \left(r \frac{\partial u}{\partial r} - u \right) \ldots\ldots (21).$$

We shall first calculate the part of this expression depending on the dilatation, for which, omitting the constant and the time-factor,

$$\left. \begin{aligned} \Delta &= \Sigma \omega_n \psi_n (hr), \\ u = -\frac{1}{h^2} \frac{\partial \Delta}{\partial x}, \quad v &= -\frac{1}{h^2} \frac{\partial \Delta}{\partial y}, \quad w = -\frac{1}{h^2} \frac{\partial \Delta}{\partial z} \end{aligned} \right\} \ldots\ldots (22).$$

Using the identity

$$xf (x, y, z) = \frac{r^2}{2n+1} \left\{ \frac{\partial f}{\partial x} - r^{2n+1} \frac{\partial}{\partial x} \left(\frac{f}{r^{2n+1}} \right) \right\} \ldots\ldots (23),$$

we find

$$x\Delta = r^2\Sigma\left[\frac{\psi_n(hr)}{2n+1}\left\{\frac{\partial\omega_n}{\partial x} - r^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\}\right] \quad \ldots\ldots\ldots(24).$$

The terms of $ux+vy+wz$ depending on Δ are $-\frac{r}{h^2}\frac{\partial\Delta}{\partial r}$

or

$$-\frac{1}{h^2}\Sigma\omega_n\left\{n\psi_n(hr) + r\frac{d\psi_n(hr)}{dr}\right\} \quad \ldots\ldots\ldots\ldots(25).$$

The terms contributed hereby to $\frac{\partial}{\partial x}(ux+vy+wz)$ are

$$-\frac{1}{h^2}\Sigma\left[\frac{\partial\omega_n}{\partial x}\left\{n\psi_n(hr) + r\frac{d\psi_n(hr)}{dr}\right\}\right.$$
$$\left. + \frac{x\omega_n}{r}\left\{(n+1)\frac{d\psi_n(hr)}{dr} + r\frac{d^2\psi_n(hr)}{dr^2}\right\}\right] \quad \ldots\ldots\ldots(26),$$

which, by using the identity (23), and the differential equation (14), become

$$-\frac{1}{h^2}\Sigma\left[\left\{n\psi_n(hr) - \frac{h^2r^2}{2n+1}\psi_n(hr) + \frac{n}{2n+1}r\frac{d\psi_n(hr)}{dr}\right\}\frac{\partial\omega_n}{\partial x}\right.$$
$$\left. + \left\{h^2\psi_n(hr) + \frac{n+1}{r}\frac{d\psi_n(hr)}{dr}\right\}\frac{r^{2n+3}}{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right] \quad \ldots\ldots(27).$$

The terms of $r\frac{\partial u}{\partial r} - u$, depending on Δ, are

$$-\frac{1}{h^2}\left(r\frac{\partial}{\partial r} - 1\right)\Sigma\left[\left\{\psi_n(hr) + \frac{1}{2n+1}r\frac{d\psi_n(hr)}{dr}\right\}\frac{\partial\omega_n}{\partial x}\right.$$
$$\left. - \frac{r^{2n+2}}{2n+1}\frac{d\psi_n(hr)}{dr}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right] \quad \ldots\ldots\ldots(28),$$

which, by using the differential equation (14) to eliminate $\frac{d^2\psi_n(hr)}{dr^2}$, become

$$-\frac{1}{h^2}\Sigma\left[\left\{(n-2)\psi_n(hr) - \frac{h^2r^2}{2n+1}\psi_n(hr) + \frac{n-2}{2n+1}r\frac{d\psi_n(hr)}{dr}\right\}\frac{\partial\omega_n}{\partial x}\right.$$
$$\left. + \left\{h^2\psi_n(hr) + \frac{n+3}{r}\frac{d\psi_n(hr)}{dr}\right\}\frac{r^{2n+3}}{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right] \quad \ldots\ldots\ldots(29).$$

195. Surface-tractions independent of Dilatation.

We next calculate the part of the expression (21) for which $\Delta = 0$ and u, v, w are such expressions as

$$\Sigma\psi_n(\kappa r)\left(y\frac{\partial\chi_n}{\partial z} - z\frac{\partial\chi_n}{\partial y}\right)$$
$$+\Sigma\left[\psi_{n-1}(\kappa r)\frac{\partial\phi_n}{\partial x} - \psi_{n+1}(\kappa r)\frac{n}{n+1}\frac{\kappa^2r^{2n+3}}{(2n+1)(2n+3)}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right].$$

The terms of $ux+vy+wz$ not depending on Δ are

$$\Sigma\left[n\psi_{n-1}(\kappa r)\,\phi_n+n\psi_{n+1}(\kappa r)\,\frac{\kappa^2r^2}{(2n+1)(2n+3)}\phi_n\right],$$

which, by (13), reduce to

$$\Sigma n\psi_n(\kappa r)\,\phi_n.$$

The terms contributed hereby to $\dfrac{\partial}{\partial x}(ux+vy+wz)$ are

$$\Sigma\left[n\left\{\psi_n(\kappa r)+\frac{r}{2n+1}\frac{d\psi_n(\kappa r)}{dr}\right\}\frac{\partial\phi_n}{dx}-\frac{n}{2n+1}r^{2n+2}\frac{d\psi_n(\kappa r)}{dr}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right]$$

.....(30).

The terms of $r\dfrac{\partial u}{\partial r}-u$ not depending on Δ are

$$\Sigma\left[\left\{(n-1)\psi_n(\kappa r)+r\frac{d\psi_n(\kappa r)}{dr}\right\}\left(y\frac{\partial\chi_n}{\partial z}-z\frac{\partial\chi_n}{\partial y}\right)\right.$$

$$+\left\{(n-2)\psi_{n-1}(\kappa r)+r\frac{d\psi_{n-1}(\kappa r)}{dr}\right\}\frac{\partial\phi_n}{\partial x}$$

$$\left.-\frac{n}{n+1}\left\{n\psi_{n+1}(\kappa r)+r\frac{d\psi_{n+1}(\kappa r)}{dr}\right\}\frac{\kappa^2r^{2n+3}}{(2n+1)(2n+3)}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right]\text{...(31).}$$

196. Formation of the boundary-conditions.

The surface $r=a$ being free from stress, we have to form three such equations as

$$\frac{\lambda}{\mu}x\Delta+\frac{\partial}{\partial x}(ux+vy+wz)+r\frac{\partial u}{\partial r}-u=0$$

when $r=a$. This equation can be formed by adding together the terms of (24) multiplied by $\dfrac{\lambda}{\mu}$, (27), (29), (30), and (31), and equating the sum to zero. The equation obtained can be written

$$\Sigma\left[p_n\left(y\frac{\partial\chi_n}{\partial z}-z\frac{\partial\chi_n}{\partial y}\right)+a_n\frac{\partial\omega_n}{\partial x}+b_nr^{2n+3}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right.$$

$$\left.+c_n\frac{\partial\phi_n}{\partial x}+d_nr^{2n+3}\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right]=0\text{.....(32),}$$

when $r=a$.

We find, after a few reductions by means of the equations (13),

and remembering that $\lambda/\mu = -2 + \kappa^2/h^2$, the following values of the coefficients

$$\left.\begin{aligned} p_n &= (n-1)\psi_n(\kappa a) + \kappa a \psi_n'(\kappa a), \\ a_n &= \frac{1}{h^2}\left[\frac{\kappa^2 a^2}{2n+1}\psi_n(ha) - 2(n-1)\psi_{n-1}(ha)\right], \\ b_n &= -\frac{1}{h^2}\frac{\kappa^2}{2n+1}\left[\psi_n(ha) + \frac{2(n+2)}{\kappa^2 a^2} ha\psi_n'(ha)\right], \\ c_n &= -\left[\frac{\kappa^2 a^2}{2n+1}\psi_n(\kappa a) - 2(n-1)\psi_{n-1}(\kappa a)\right], \\ d_n &= -\frac{n}{n+1}\frac{\kappa^2}{2n+1}\left[\psi_n(\kappa a) + \frac{2(n+2)}{\kappa^2 a^2}\kappa a\,\psi_n'(\kappa a)\right] \end{aligned}\right\} \quad \ldots(33).$$

in which $\psi_n'(x)$ is $\dfrac{d\psi_n(x)}{dx}$.

197. Formation of the frequency-equations.

There are three such equations as (32), which hold when $r = a$. The left hand sides of these are finite, continuous, and one-valued within the sphere $r = a$, they satisfy Laplace's equation, and vanish at the surface. They are therefore identically zero. In equation (32) and the like equations we may suppress the sign of summation, and the equations thus obtained hold for each value of n and for all values of r.

Differentiating these equations with respect to x, y, z and adding, we have

$$b_n\omega_n + d_n\phi_n = 0 \quad \ldots\ldots(34).$$

Multiplying these equations by x, y, z, adding, and using the equation just found, we have

$$a_n\omega_n + c_n\phi_n = 0 \quad \ldots\ldots(35).$$

Using (34) and (35) in the equation obtained from (32), we have

$$p_n = 0 \quad \ldots\ldots(36).$$

Now $p_n = 0$ is an equation involving only κ, a, and the number n; κ depends only on the frequency $p/2\pi$, the rigidity μ, and the density ρ. Thus $p_n = 0$ is a frequency-equation.

In like manner the equation

$$a_n d_n - b_n c_n = 0 \quad \ldots\ldots(37)$$

obtained by eliminating ω_n, ϕ_n from equations (34) and (35) is a frequency-equation.

198. Vibrations of the First Class.

We now see that the vibrations fall naturally into two classes. For the first of these χ_n is the only harmonic that occurs, $\Delta = 0$, and $ux + vy + wz = 0$, so that the motion is purely tangential. The frequency-equation is $p_n = 0$, or

$$(n-1)\psi_n(\kappa a) + \kappa a \psi_n'(\kappa a) = 0.$$

Prof. Lamb gives an account of the simpler cases. We shall follow his description of the different species of vibrations.

Species $n = 1$. *Rotatory Vibrations.*

If we take the axis of the harmonic χ_1 as axis of z, we shall get for the normal functions

$$u = \psi_1(\kappa r)\, y, \quad v = -\psi_1(\kappa r)\, x, \quad w = 0.$$

Each of the infinitely thin concentric spherical strata of which the sphere may be supposed built up turns round the axis of z through a small angle proportional to $\psi_1(\kappa r)$. The frequency-equation is $\psi_1'(\kappa a) = 0$, and this may be written

$$\tan \kappa a = \frac{3\kappa a}{3 - \kappa^2 a^2}.$$

The first six roots of this equation are[1]

$$\frac{\kappa a}{\pi} = 1{\cdot}8346,\ 2{\cdot}8950,\ 3{\cdot}9225,\ 4{\cdot}9385,\ 5{\cdot}9489,\ 6{\cdot}9563.$$

The number $\pi/\kappa a$ is equal to the ratio of the period of oscillation to the time taken by a wave of distortion to travel a distance equal to the diameter of the sphere. In any mode, after the first, the roots of lower order give the positions of the spherical loop surfaces (where the radial stress vanishes). Thus for the second mode there is a loop given by $r = {\cdot}6337a$. The positions of the spherical nodes are given by $\psi_1(\kappa r) = 0$, or $\tan \kappa r = \kappa r$ and the first six roots of this are

$$\kappa r/\pi = 1{\cdot}4303,\ 2{\cdot}4590,\ 3{\cdot}4709,\ 4{\cdot}4774,\ 5{\cdot}4818,\ 6{\cdot}4844.$$

Species $n = 2$.

The frequency-equation is

$$\psi_2(\kappa a) + \kappa a \psi_2'(\kappa a) = 0,$$

which may be written

$$\frac{\tan \kappa a}{\kappa a} = \frac{12 - \kappa^2 a^2}{12 - 5\kappa^2 a^2}.$$

[1] For the analysis by which this and the similar results in the present and the following article are reached the reader is referred to Prof. Lamb's paper in *Proc. Lond. Math. Soc.* XIII., 1882.

The first six roots are found to be

$$\kappa a/\pi = \cdot 7961,\ 2\cdot 2715,\ 3\cdot 3469,\ 4\cdot 3837,\ 5\cdot 4059,\ 6\cdot 4209.$$

The character of the vibration depends on the form of χ_2. In the case where χ_2 is the zonal harmonic $2z^2 - x^2 - y^2$, we have for the normal functions

$$u = \psi_2(\kappa r)\, yz, \quad v = -\psi_2(\kappa r)\, xz, \quad w = 0.$$

All the particles on the same parallel move along the parallel through a small distance proportional to the sine of the latitude, and the equatorial plane is nodal.

199. Vibrations of the Second Class.

For these χ_n is zero, and the harmonics that occur are ω_n and ϕ_n, and we shall find that in general the motion is partly radial and partly tangential. The frequency-equation is

$$b_n c_n - a_n d_n = 0,$$

where a_n, b_n, c_n, d_n are given by (33). It will be seen that in general both h and κ occur in this equation, and therefore its solution cannot be reduced to a question of arithmetic until the ratio of the elastic constants λ and μ is given. In general we shall consider two cases (1) where the material is incompressible, or λ is very great compared with μ, for which h is very small compared with κ; and (2) where $\lambda = \mu$, (Poisson's condition,) for which $\kappa = \sqrt{3}h$.

Species $n = 0$. *Radial Vibrations.*

For these the normal functions are

$$u = -\frac{1}{h^2}\frac{x}{r}\psi_0'(hr), \quad v = -\frac{1}{h^2}\frac{y}{r}\psi_0'(hr), \quad w = -\frac{1}{h^2}\frac{z}{r}\psi_0'(hr),$$

and the frequency-equation becomes simply $b_0 = 0$, or

$$\psi_0(ha) + \frac{4}{\kappa^2 a^2} ha\psi_0'(ha) = 0.$$

This is $$\tan ha = 4ha \Big/ \left(4 - \frac{\kappa^2}{h^2} h^2 a^2\right).$$

When $\lambda = \mu$, this becomes

$$(\tan ha)/ha = 1/(1 - \tfrac{3}{4} h^2 a^2)$$

and the first six roots are given by

$$ha/\pi = \cdot 8160,\ 1\cdot 9285,\ 2\cdot 9359,\ 3\cdot 9658,\ 4\cdot 9728,\ 5\cdot 9774.$$

The number π/ha is the ratio of the period of oscillation to the time taken by a wave of compression to travel a distance equal to the diameter of the sphere.

For the higher modes of vibration the roots of lower order give the position of the spherical loop-surfaces across which there is no stress. The spherical nodes are given by $\psi_0'(hr) = 0$, or $\tan hr = hr$ and the roots of this are given in art. 198. It appears that, in the sth mode, there are $s-1$ nodal spheres at which the displacement vanishes. The theory of the free radial vibrations is an interesting example of the general theory of those classes of vibrations, for which the displacement of any point can be expressed by means of a single function. This is the class of cases treated in Lord Rayleigh's *Theory of Sound*, arts. 93, 94, and 101. The displacement ζ of those articles can be taken to be the radial displacement of any point within the sphere, and is given by an equation of the form

$$\zeta = u_1\phi_1 + u_2\phi_2 + \ldots,$$

where $u_1, u_2, \ldots$ are the normal functions, and $\phi_1, \phi_2, \ldots$ are the normal coordinates. Suppose $h_1, h_2, \ldots$ are the values of h obtained from the frequency-equation, and $p_1/2\pi, p_2/2\pi, \ldots$ the corresponding frequencies. Then the normal coordinates $\phi_1, \phi_2, \ldots$ are identical with quantities of the form

$$A_1 \cos(p_1 t + \epsilon_1), \quad A_2 \cos(p_2 t + \epsilon_2), \ldots$$

where $A_1, A_2, \ldots$ and $\epsilon_1, \epsilon_2, \ldots$ are arbitrary constants. The normal functions $u_1, u_2, \ldots$ are given by

$$u_1 = \psi_0'(h_1 r), \quad u_2 = \psi_0'(h_2 r), \ldots$$

Species $n = 1$. *Incompressible material.*

The frequency-equation reduces to

$$\kappa a \psi_1(\kappa a) + 2\psi_1'(\kappa a) = 0,$$

or

$$\frac{\tan \kappa a}{\kappa a} = \frac{6 - \kappa^2 a^2}{6 - 3\kappa^2 a^2},$$

and the first six roots are given by

$$\kappa a/\pi = 1{\cdot}2319,\ 2{\cdot}3692,\ 3{\cdot}4101,\ 4{\cdot}4310,\ 5{\cdot}4439,\ 6{\cdot}4528.$$

We may take $\phi_1 = z$, and then equation (35) becomes

$$\omega_1 = h^2 \frac{\psi_1(\kappa a)}{\psi_1(ha)} z,$$

so that ω_1 may be taken $= 0$, but ω_1/h^2 is not $= 0$. The radial

displacement at any point is proportional to $r^{-1}\{\psi_1(\kappa r) - \psi_1(\kappa a)\}\, z$, so that in the sth mode there are $s-1$ spherical surfaces at which the radial displacement vanishes. We may term these surfaces "quasi-nodal", and the equatorial plane is in like manner a quasi-nodal surface.

Species $n = 1$. *Material fulfilling Poisson's condition.*

Equations (34) and (35) become

$$\psi_1(ha)\frac{\omega_1}{h^2} - \psi_1(\kappa a)\,\phi_1 = 0,$$

$$\left\{\psi_1(ha) + \frac{6}{\kappa^2 a^2} ha\psi_1'(ha)\right\}\frac{\omega_1}{h^2} + \tfrac{1}{2}\left\{\psi_1(\kappa a) + \frac{6}{\kappa a}\psi_1'(\kappa a)\right\}\phi_1 = 0,$$

and the frequency-equation, obtained by eliminating ω_1/h^2 and ϕ_1, and supposing $\kappa = \sqrt{3}h$, is

$$1 + \frac{\frac{4}{\sqrt{3}}\psi_1'\left(\frac{\kappa a}{\sqrt{3}}\right)}{\kappa a\psi_1\left(\frac{\kappa a}{\sqrt{3}}\right)} + \frac{2\psi_1'(\kappa a)}{\kappa a\psi_1(\kappa a)} = 0.$$

The first three roots can be shewn to be

$$\kappa a/\pi = 1{\cdot}090, \quad 2{\cdot}155, \quad 2{\cdot}465, \ldots$$

The radial displacement is proportional to

$$\frac{z}{r}\left[\psi_1(\kappa r) - \frac{\psi_1(\kappa a)}{\psi_1(ha)}\{\psi_1(hr) + hr\psi_1'(hr)\}\right],$$

z being written for ϕ_1, and the quasi-nodal spherical surfaces are found by equating the function in square brackets to zero. The radial displacement is finite at the free surface, and it can be shewn that, for the second mode of vibration, there exists one internal quasi-nodal spherical surface. In general for the sth mode there do not exist so many as $s-1$ of these surfaces.

Species $n = 2$.

Equations (34) and (35) become

$$\left\{\frac{\kappa^2 a^2}{5}\psi_2(ha) - 2\psi_1(ha)\right\}\frac{\omega_2}{h^2} - \left\{\frac{\kappa^2 a^2}{5}\psi_2(\kappa a) - 2\psi_1(\kappa a)\right\}\phi_2 = 0,$$

$$\left\{\psi_2(ha) + \frac{8}{\kappa^2 a^2} ha\psi_2'(ha)\right\}\frac{\omega_2}{h^2} + \tfrac{2}{3}\left\{\psi_2(\kappa a) + \frac{8}{\kappa^2 a^2}\kappa a\psi_2'(\kappa a)\right\}\phi_2 = 0.$$

The lowest root of the equation for $\kappa a/\pi$ found by eliminating

ω_2/h^2 and ϕ_2, when κ is great compared with h or the material is incompressible, is ·848, and, when $\kappa = \sqrt{3}h$ or the material fulfils Poisson's condition, ·840.

For a sphere of the size and mass of the earth supposed incompressible, and as rigid as steel or iron, (art. 184), the period of the gravest free vibration, in which the surface becomes a harmonic spheroid of the second order, is about 1 hr. 6 min. If it be as rigid as glass the period is 2 hrs. nearly.

200. Vibrations of a spherical shell.

In case the vibrating solid is bounded by two concentric spheres we shall have to introduce the second solution $\Psi_n(x)$ of the differential equation (14) of art. 192. The equations of motion are equations (1) of art. 191, and these lead, just as in that article, to equations (5) and (8).

The complete solution of (5) for space between two concentric spheres is

$$\Delta = \Sigma\left[\omega_n \psi_n(hr) + \Omega_n \Psi_n(hr)\right] \ldots\ldots\ldots\ldots\ldots\ldots (38),$$

where ω_n and Ω_n are spherical solid harmonics, and $\psi_n(hr)$ and $\Psi_n(hr)$ are defined by the equations

$$\left.\begin{aligned} \psi_n(x) &= (-)^n\, 1\,.\,3 \ldots (2n+1)\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right) \\ \Psi_n(x) &= (-)^n\, 1\,.\,3 \ldots (2n+1)\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\cos x}{x}\right) \end{aligned}\right\} \ldots (39).$$

Both these functions are finite, continuous, and one-valued for the space considered, and they satisfy the same differential equation, the same difference-equation, and the same mixed difference-equations. These equations are (14) and (13) of art. 192. The function $\Psi_n(x)$ is connected with Bessel's function of the second kind by a relation of the form

$$x^{n+\frac{1}{2}}\Psi_n(x) = A\, Y_{n+\frac{1}{2}}(x),$$

where the constant A depends on the form assumed for the Bessel's function. This function $\Psi_n(x)$ has two critical points, the origin and the point at infinity. The first is a pole of the $(2n+1)$th order, *i.e.* the product $x^{2n+1}\Psi_n(x)$ has a finite limit when $x = 0$, the second is an essential critical point of the function.

With the same notation we can write down the general

solutions of the equations of vibration for an isotropic homogeneous solid bounded by two concentric spheres; we have, just as in art. 193,

$$u = -\frac{1}{h^2}\frac{\partial \Delta}{\partial x}$$
$$+ \Sigma\left[\psi_n(\kappa r)\left(\frac{\partial \phi_{n+1}}{\partial x} + y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y}\right)\right.$$
$$\left. - \frac{n+1}{n+2}\frac{\kappa^2 r^{2n+5}}{(2n+3)(2n+5)}\psi_{n+2}(\kappa r)\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right)\right]$$
$$+ \Sigma\left[\Psi_n(\kappa r)\left(\frac{\partial \Phi_{n+1}}{\partial x} + y\frac{\partial \mathrm{X}_n}{\partial z} - z\frac{\partial \mathrm{X}_n}{\partial y}\right)\right.$$
$$\left. - \frac{n+1}{n+2}\frac{\kappa^2 r^{2n+5}}{(2n+3)(2n+5)}\Psi_{n+2}(\kappa r)\frac{\partial}{\partial x}\left(\frac{\Phi_{n+1}}{r^{2n+3}}\right)\right] \ldots\ldots\ldots(40),$$

where Δ is given by (38), and χ_n, X_n, ϕ_{n+1}, Φ_{n+1} are spherical solid harmonics whose orders are indicated by the suffixes.

The boundary-conditions can be obtained just as in arts. 194—196, and they can be written in the form

$$\Sigma\left[p_n\left(y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y}\right) + a_n\frac{\partial \omega_n}{\partial x} + b_n\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right) + c_n\frac{\partial \phi_n}{\partial x} + d_n\frac{\partial}{\partial x}\left(\frac{\phi_n}{r^{2n+1}}\right)\right.$$
$$+ P_n\left(y\frac{\partial \mathrm{X}_n}{\partial z} - z\frac{\partial \mathrm{X}_n}{\partial y}\right) + A_n\frac{\partial \Omega_n}{\partial x} + B_n\frac{\partial}{\partial x}\left(\frac{\Omega_n}{r^{2n+1}}\right)$$
$$\left. + C_n\frac{\partial \Phi_n}{\partial x} + D_n\frac{\partial}{\partial x}\left(\frac{\Phi_n}{r^{2n+1}}\right)\right] = 0 \ldots\ldots\ldots(41),$$

where p_n, a_n... are the functions of a given in equations (33), and P_n, A_n... are the same functions with Ψ's in place of ψ's. There are six equations such as (41). Of these two are obtained from (41) by cyclical interchanges of the letters x, y, z, and the other three are obtained from (41) by putting b for a, b being the radius of the outer surface and a that of the inner. If p_n', a_n'...P_n', A_n'... denote the same functions of b that p_n, a_n...P_n, A_n... are of a, we can deduce from these, by the process of art. 197, the following conditions

$$\left.\begin{aligned} p_n\left(y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y}\right) + P_n\left(y\frac{\partial \mathrm{X}_n}{\partial z} - z\frac{\partial \mathrm{X}_n}{\partial y}\right) = 0 \\ p_n'\left(y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y}\right) + P_n'\left(y\frac{\partial \mathrm{X}_n}{\partial z} - z\frac{\partial \mathrm{X}_n}{\partial y}\right) = 0 \end{aligned}\right\} \ldots\ldots(42),$$

and

$$\left.\begin{aligned} a_n\omega_n + c_n\phi_n + A_n\Omega_n + C_n\Phi_n &= 0 \\ b_n\omega_n + d_n\phi_n + B_n\Omega_n + D_n\Phi_n &= 0 \\ a_n'\omega_n + c_n'\phi_n + A_n'\Omega_n + C_n'\Phi_n &= 0 \\ b_n'\omega_n + d_n'\phi_n + B_n'\Omega_n + D_n'\Phi_n &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(43).$$

From (42) we find

$$p_nP_n' - p_n'P_n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (44),$$

and from (43) we find

$$\begin{vmatrix} a_n, & c_n, & A_n, & C_n \\ b_n, & d_n, & B_n, & D_n \\ a_n', & c_n', & A_n', & C_n' \\ b_n', & d_n', & B_n', & D_n' \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots \quad (45).$$

These are the frequency-equations.

For the particular case of an indefinitely thin shell we have to put $b = a + \delta a$, and then the second equation of (42) becomes

$$\frac{\partial p_n}{\partial a}\left(y\frac{\partial \chi_n}{\partial z} - z\frac{\partial \chi_n}{\partial y}\right) + \frac{\partial P_n}{\partial a}\left(y\frac{\partial \mathrm{X}_n}{\partial z} - z\frac{\partial \mathrm{X}_n}{\partial y}\right),$$

and the third and fourth of (43) become

$$\frac{\partial a_n}{\partial a}\omega_n + \frac{\partial c_n}{\partial a}\phi_n + \frac{\partial A_n}{\partial a}\Omega_n + \frac{\partial C_n}{\partial a}\Phi_n,$$

$$\frac{\partial b_n}{\partial a}\omega_n + \frac{\partial d_n}{\partial a}\phi_n + \frac{\partial B_n}{\partial a}\Omega_n + \frac{\partial D_n}{\partial a}\Phi_n.$$

The frequency-equations have the same forms as before, but the accented letters must now be regarded as the differential coefficients of the unaccented letters with respect to the radius.

It should be noticed that to a first approximation the resulting equations depend only on the radius of the shell, the elastic constants, the density, and the frequency, and are independent of the thickness. This result is of importance in the theory of thin shells. It shews that for a complete thin spherical shell all the periods of free vibration are independent of the thickness of the shell.

201. Forced vibrations of solid sphere.

We shall next consider the vibrations produced in a sphere whose surface is free by the action of periodic forces derivable from a potential expressible in spherical harmonic series.

Suppose that a single term of the series is the real part of

$$W_{n+1}\, e^{ipt},$$

where W_{n+1} is a spherical solid harmonic of degree $n+1$.

The forced vibrations will be obtained by assuming that as functions of t the displacements u, v, w are all proportional to e^{ipt}.

The equations of motion can be written in such forms as

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u=-\rho\left(p^2u+\frac{\partial W_{n+1}}{\partial x}\right)\ldots\ldots\ldots(46),$$

where u is written for the coefficient of e^{ipt} in the expression for the displacement parallel to x, and Δ for the coefficient of e^{ipt} in the expression for the cubical dilatation.

We have already in art. 139 given the particular integrals of these differential equations in the form

$$u=-\frac{1}{p^2}\frac{\partial W_{n+1}}{\partial x},\quad v=-\frac{1}{p^2}\frac{\partial W_{n+1}}{\partial y},\quad w=-\frac{1}{p^2}\frac{\partial W_{n+1}}{\partial z}\ldots(47).$$

These solutions make the cubical dilatation Δ vanish, and they give for ζ, the product of the radial displacement and the radius-vector, the expression

$$\zeta=ux+vy+wz=-\frac{n+1}{p^2}W_{n+1}.$$

The surface-tractions hence arising are easily shewn to be given by such equations as

$$Fr=-\frac{2n\mu}{p^2}\frac{\partial W_{n+1}}{\partial x}\ldots\ldots\ldots\ldots\ldots\ldots(48),$$

omitting the time-factor.

For the complementary solutions we shall assume the forms

$$u=-\frac{1}{h^2}\frac{\partial}{\partial x}\Sigma\omega_{n+1}\psi_{n+1}(hr)$$
$$+\Sigma\left[\psi_{n+1}(\kappa r)\frac{\partial\phi_{n+1}}{\partial x}-\frac{n+1}{n+2}\frac{\kappa^2r^{2n+5}}{(2n+3)(2n+5)}\psi_{n+2}(\kappa r)\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right)\right],$$

omitting the χ terms from the general solution. The vibrations depending on these terms would not be forced by the actions considered.

The surface-tractions arising from the complementary solutions are known to be given by such equations as

$$Fr=\mu\Sigma\left[a_{n+1}\frac{\partial\omega_{n+1}}{\partial x}+b_{n+1}r^{2n+5}\frac{\partial}{\partial x}\left(\frac{\omega_{n+1}}{r^{2n+3}}\right)+c_{n+1}\frac{\partial\phi_{n+1}}{\partial x}\right.$$
$$\left.+d_{n+1}r^{2n+5}\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right)\right]\ldots\ldots\ldots\ldots(49).$$

To get the boundary-conditions we have simply to add the parts of Fr arising from the complementary solutions and the particular integrals and equate the result to zero; we find in this way three equations of which the type is

$$\Sigma\left[a_{n+1}\frac{\partial\omega_{n+1}}{\partial x}+b_{n+1}r^{2n+5}\frac{\partial}{\partial x}\left(\frac{\omega_{n+1}}{r^{2n+3}}\right)+c_{n+1}\frac{\partial\phi_{n+1}}{\partial x}+d_{n+1}r^{2n+5}\frac{\partial}{\partial x}\left(\frac{\phi_{n+1}}{r^{2n+3}}\right)\right]$$

$$=\frac{2n}{p^2}\frac{\partial W_{n+1}}{\partial x}\quad\ldots\ldots\ldots\ldots(50).$$

Now, operating upon these equations in the same way as in art. 197 upon the equations of the form (32), we obtain the following:

$$\left.\begin{aligned} b_{n+1}\omega_{n+1}+d_{n+1}\phi_{n+1}&=0\\ a_{n+1}\omega_{n+1}+c_{n+1}\phi_{n+1}&=\frac{2n}{p^2}W_{n+1}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(51),$$

which give

$$\frac{\omega_{n+1}}{d_{n+1}}=\frac{\phi_{n+1}}{-b_{n+1}}=\frac{2n}{p^2}\frac{W_{n+1}}{a_{n+1}d_{n+1}-b_{n+1}c_{n+1}}\quad\ldots\ldots\ldots(52).$$

These equations determine the unknown harmonics ω_{n+1} and ϕ_{n+1} that occur in the complementary solutions, and they shew that to each term W_{n+1} of the disturbing potential there corresponds one function ω and one function ϕ.

It is easy to shew that the height of the harmonic inequality is

$$\frac{1}{a}\left[(n+1)\,\psi_{n+1}(\kappa a)\,\phi_{n+1}\right.$$

$$\left.-\left\{(n+1)\,\psi_{n+1}(ha)+a\,\frac{d\psi_{n+1}(ha)}{da}\right\}\frac{\omega_{n+1}}{h^2}-\frac{n+1}{p^2}W_{n+1}\right]e^{ipt}\ldots(53),$$

so that this is of the form $\frac{\epsilon_n W_{n+1}}{ap^2}e^{ipt}$, where ϵ_n is a number; and the equation of the surface of the sphere at time t is

$$r=a+\frac{\epsilon_n W_{n+1}}{ap^2}e^{ipt}\quad\ldots\ldots\ldots\ldots\ldots(54).$$

202. Disturbing Potential a spherical harmonic of the second order.

The case $n=1$ is the most interesting. In this case the disturbing potential is a spherical solid harmonic of order 2. We have

$$\left.\begin{aligned} a_2 &= \frac{1}{h^2}\left\{\frac{\kappa^2 a^2}{5}\psi_2(ha) - 2\psi_1(ha)\right\}, \\ b_2 &= -\frac{1}{h^2}\left\{\frac{\kappa^2}{5}\psi_2(ha) + \frac{8}{\kappa^2 a^2}ha\psi_2'(ha)\right\}, \\ c_2 &= -\left\{\frac{\kappa^2 a^2}{5}\psi_2(\kappa a) - 2\psi_1(\kappa a)\right\}, \\ d_2 &= -\frac{2\kappa^2}{15}\left\{\psi_2(\kappa a) + \frac{8}{\kappa^2 a^2}\kappa a\psi_2'(\kappa a)\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots(55).$$

The height of the harmonic inequality is

$$\frac{W_2}{ap^2}\left[2\,\frac{2\psi_2(\kappa a)\,b_2 + \{2\psi_2(ha) + ha\psi_2'(ha)\}\,d_2/h^2}{b_2c_2 - a_2d_2} - 2\right]e^{ipt} \ldots(56).$$

We give the arithmetical result for the special case when the material is incompressible and so rigid that the fourth power of κa may be neglected.

We have by (15)

$$ha = 0, \quad \psi_1(ha) = 1, \quad \psi_2(ha) = 1, \quad \psi_2'(ha) = 0,$$

$$\psi_1(\kappa a) = 1 - \frac{\kappa^2 a^2}{10}, \quad \psi_2(\kappa a) = 1 - \frac{\kappa^2 a^2}{14} + \frac{\kappa^4 a^4}{504}.$$

Thus the constants are given by

$$a_2 = \frac{1}{h^2}\left(\frac{\kappa^2 a^2}{5} - 2\right), \quad b_2 = -\frac{1}{h^2}\frac{\kappa^2}{5},$$

$$c_2 = 2\left(1 - \frac{\kappa^2 a^2}{5}\right),$$

$$d_2 = -\frac{2\kappa^2}{15}\left\{1 - \frac{\kappa^2 a^2}{14} + \frac{8}{\kappa^2 a^2}\left(-\frac{\kappa^2 a^2}{7} + \frac{\kappa^4 a^4}{126}\right)\right\}.$$

The height of the harmonic inequality becomes, when we reject $\kappa^4 a^4$ in the numerator and denominator of (56),

$$\frac{2W_2}{ap^2}\left[-1 + \frac{-\frac{38}{21} + \kappa^2 a^2 \frac{29}{189}}{-\frac{38}{21} + \kappa^2 a^2 \frac{74}{189}}\right]e^{ipt}.$$

Simplifying, we get for the height of the harmonic inequality

$$\tfrac{5}{19}\frac{\rho a}{\mu}W_2 e^{ipt} \ldots\ldots\ldots\ldots\ldots\ldots(57),$$

which agrees with the result of the corresponding equilibrium problem in art. 182.

For a sphere of the mass and diameter of the earth, and of the rigidity of steel or iron, executing vibrations of the species considered with a semi-diurnal period, we have, in C.G.S. units

$$2\pi/p = 12 \times 60 \times 60,\ \rho = 5{\cdot}6,\ \mu = 800 \times 10^9,\ a = 640 \times 10^6,$$

so that $\kappa a = \frac{1}{4}$ nearly.

It follows from this that the neglect of $(\kappa a)^4$ would be fairly justifiable in the case of such a body. We conclude that in the case of an elastic solid earth the bodily tides would follow the equilibrium law.

203. Plane Waves propagated at the surface of an elastic solid.

Another extension of Prof. Lamb's analysis is that of Lord Rayleigh[1], who has applied a similar method to discuss waves propagated at the plane surface of an elastic solid, the disturbance being practically confined to a superficial region comparable with the wave-length. We give an account of Lord Rayleigh's method and results.

The differential equations of the problem are the same as those established in art. 191. Taking $z = 0$ for the free surface, and z positive within the solid, we suppose that, as functions of x and y, the displacements are proportional to $e^{\iota(fx+gy)}$. Then (5) takes the form

$$\frac{\partial^2 \Delta}{\partial z^2} - r^2 \Delta = 0 \quad \text{.....(58),}$$

where

$$r^2 = f^2 + g^2 - h^2 \quad \text{.....(59).}$$

Supposing r real and positive, we have

$$\Delta = Pe^{-rz} \quad \text{.....(60),}$$

where $P \propto e^{\iota(fx+gy+pt)}$; and the particular solutions (7) become

$$u = -\frac{\iota f}{h^2} Pe^{-rz},\ v = -\frac{\iota g}{h^2} Pe^{-rz},\ w = \frac{r}{h^2} Pe^{-rz} \quad \text{.....(61).}$$

The complementary solutions satisfy equations such as

$$\frac{\partial^2 u}{\partial z^2} - s^2 u = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{.....(62),}$$

where

$$s^2 = f^2 + g^2 - \kappa^2 \quad \text{.....(63);}$$

[1] *Proc. Lond. Math. Soc.* XVII. 1886.

so that we have

$$u = Ae^{-sz}, \qquad v = Be^{-sz}, \qquad w = Ce^{-sz} \ldots\ldots\ldots (64),$$

where

$$\iota fA + \iota gB - sC = 0 \ldots\ldots\ldots\ldots\ldots\ldots (65),$$

and A, B, C contain a factor $e^{\iota(fx+gy+pt)}$.

Hence the expressions for the displacements

$$u = -\frac{\iota f}{h^2} Pe^{-rz} + Ae^{-sz}, \quad v = -\frac{\iota g}{h^2} Pe^{-rz} + Be^{-sz}, \quad w = \frac{r}{h^2} Pe^{-rz} + Ce^{-sz}$$
$$\ldots\ldots\ldots (66).$$

The boundary-conditions and equation (65) give, by taking $P = 1$,

$$\left.\begin{aligned} sB = \frac{2\iota gr}{h^2} + \iota gC, \quad sA = \frac{2\iota fr}{h^2} + \iota fC, \\ C(s^2 + f^2 + g^2)h^2 + 2r(f^2 + g^2) = 0, \\ \kappa^2 - 2(f^2 + g^2) - 2h^2 sC = 0 \end{aligned}\right\} \ldots\ldots\ldots (67);$$

and the frequency-equation is

$$\kappa'^8 - 8\kappa'^6 + 24\kappa'^4 - 16\kappa'^2 - 16\kappa'^2 h'^2 + 16h'^2 = 0 \ldots\ldots (68),$$

where

$$\kappa'^2 = \kappa^2/(f^2 + g^2), \text{ and } h'^2 = h^2/(f^2 + g^2) \ldots\ldots\ldots (69).$$

When the solid is incompressible $h'^2 = 0$, and the equation for κ'^2 viz.:

$$\kappa'^6 - 8\kappa'^4 + 24\kappa'^2 - 16 = 0 \ldots\ldots\ldots\ldots (70),$$

has one real root $\kappa' = \cdot 91275$, while the complex roots make the real parts of r and s have opposite signs, so that they may be rejected. We now have

$$\kappa^2 = \cdot 91275\,(f^2 + g^2), \quad r^2 = f^2 + g^2, \quad s^2 = \cdot 08725\,(f^2 + g^2),$$

and

$$\left.\begin{aligned} h^2 u = \iota f(-e^{-rz} + \cdot 5433e^{-sz})\, e^{\iota(pt+fx+gy)} \\ h^2 v = \iota g(-e^{-rz} + \cdot 5433e^{-sz})\, e^{\iota(pt+fx+gy)} \\ h^2 w = \sqrt{(f^2 + g^2)}(e^{-rz} - 1\cdot 840e^{-sz})\, e^{\iota(pt+fx+gy)} \end{aligned}\right\} \ldots\ldots (71).$$

For progressive waves whose fronts are parallel to the axis of y, we have

$$\left.\begin{aligned} u = U(e^{-fz} - \cdot 5433e^{-sz}) \sin(pt + fx), \\ w = U(e^{-fz} - 1\cdot 840e^{-sz}) \cos(pt + fx) \end{aligned}\right\} \ldots\ldots\ldots (72),$$

where U is a constant; and the velocity of propagation is

$$p/f = \cdot 9554\sqrt{(\mu/\rho)},$$

which is slightly less than that of waves of distortion in an

unlimited medium. The horizontal motion vanishes at a certain depth. The motion at the surface is given by

$$\left.\begin{aligned} u &= \cdot 4567\, U \sin(pt + fx) \\ w &= -\cdot 840\ U \cos(pt + fx) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (73);$$

so that the particles move in elliptic orbits whose axes are nearly in the ratio 2 : 1.

Lord Rayleigh also considers the cases where $\lambda = \mu$, or the material fulfils Poisson's condition, where $\lambda = 0$, or longitudinal extension is unaccompanied by lateral contraction, and where $\lambda = -\frac{2}{3}\mu$, or the bulk-modulus vanishes. For $\lambda = \mu$ he finds

$$\kappa^2 = \cdot 8453\,(f^2 + g^2),\ r^2 = \cdot 7182\,(f^2 + g^2),\ s^2 = \cdot 1547\,(f^2 + g^2).$$

For a progressive wave

$$\left.\begin{aligned} u &= U(e^{-rz} - \cdot 5773 e^{-sz}) \sin(pt + fx), \\ w &= U(\cdot 8475 e^{-rz} - 1\cdot 4679 e^{-sz}) \cos(pt + fx) \end{aligned}\right\} \quad \ldots\ldots (74),$$

and the ratio of the axes of the elliptic orbit, described by a surface-particle, is reduced to about $\frac{3}{2}$.

Lord Rayleigh suggests that these surface-waves may play an important part in earthquakes and in collision, as they diverge from the source of disturbance in two dimensions only, and consequently gain increasing relative importance at a considerable distance.

CHAPTER XII.

APPLICATIONS OF CONJUGATE FUNCTIONS.

204. So far as I am aware, the only successful attempt hitherto made, to obtain general solutions of the equations of elastic equilibrium in a form adapted to satisfy arbitrary boundary-conditions at any other surface than a sphere or a plane, is that of Herr Wangerin[1]. He has shewn how to obtain solutions in terms of conjugate functions of the equations of equilibrium, under no bodily forces, for an isotropic body bounded by a surface of revolution for which Laplace's equation can be solved. We shall give a *résumé* of his results, and shall then proceed to illustrate the application of conjugate functions to problems of elastic equilibrium by solving some questions relating to plane strain.

205. Wangerin's Problem.

Consider in the first place cylindrical coordinates ϖ, ϕ, z, where z is the distance of any point from a fixed plane, ϖ the distance of the point from the axis z, and ϕ the angle between the axial plane through the point, and a fixed axial plane through the axis z. In the meridian plane (z, ϖ) suppose two systems of orthogonal curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ given by the equation

$$\alpha + \iota\beta = f(z + \iota\varpi) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

These curves being rotated about the axis z give rise to a system of orthogonal surfaces whose parameters are α, β, ϕ, and we may

[1] 'Ueber das Problem des Gleichgewichts elastischer Rotationskörper', Grunert's *Archiv*, LV. 1873.

use the formulæ of ch. VII. The h's are $h_1 = h_2 = h$ say, and $h_3 = \varpi^{-1}$, and we have

$$h^2 = \frac{\partial(\alpha, \beta)}{\partial(z, \varpi)}.$$

In Herr Wangerin's work h is replaced by J^{-1}, and J^2 is the Jacobian of z, ϖ with respect to α, β, or we have

$$J = \text{mod}\, \frac{d(z + \iota\varpi)}{d(\alpha + \iota\beta)} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Laplace's equation takes the form

$$\frac{\partial}{\partial\alpha}\left(\varpi \frac{\partial V}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\varpi \frac{\partial V}{\partial\beta}\right) + \frac{J^2 \partial^2 V}{\varpi\, \partial\phi^2} = 0 \ldots\ldots\ldots\ldots (3).$$

The solution of this equation takes different forms according as V is or is not a function of ϕ. In the first case we may suppose that so far as it depends upon ϕ it contains a factor $e^{\iota s\phi}$ where s is an integer, and we may denote a solution by $X_s e^{\iota s\phi}$, where X_s is a function of α and β. In the second case we may denote a solution by X_0.

The cubical dilatation Δ may be expressed in the form

$$\Delta = \Delta_0 + \sum_1^\infty \Delta_s e^{\iota s\phi},$$

and it is shewn that Δ_0 is a function of the same form as X_0, and Δ_s is a function of the same form as X_s. We therefore write

$$\Delta = X_0 + \sum_1^\infty X_s e^{\iota s\phi} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The three rotations ϖ_1, ϖ_2, ϖ_3 can be most simply found by putting

$$2\varpi_1 J = \Theta_1, \quad 2\varpi_2 J = \Theta_2, \quad 2\varpi_3 \varpi = \Theta_3 \ldots\ldots\ldots\ldots (5).$$

Then Θ_1, Θ_2, Θ_3 can be expressed in the forms

$$\left.\begin{aligned} \Theta_1 &= L_0 + \sum_1^\infty L_s e^{\iota s\phi} \\ \Theta_2 &= M_0 + \sum_1^\infty M_s e^{\iota s\phi} \\ \Theta_3 &= N_0 + \sum_1^\infty N_s e^{\iota s\phi} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the L's, M's and N's can be written as follows:

$$N_0 = -\frac{\lambda + 2\mu}{\mu} \int \varpi \frac{\partial X_0}{\partial\beta} d\alpha - \varpi \frac{\partial X_0}{\partial\alpha} d\beta \ldots\ldots\ldots\ldots (7),$$

which is the integral of a complete differential in virtue of the equation satisfied by X_0, also

$$L_0 = \frac{\partial Y_0}{\partial \alpha}, \qquad M_0 = \frac{\partial Y_0}{\partial \beta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

where Y_0 is a function of the same form as X_0, *i.e.* a solution independent of ϕ of equation (3).

Also we have

$$\left.\begin{aligned} N_s &= Y_s + \frac{\lambda+2\mu}{\mu} z X_s \\ \iota s L_s &= \frac{\partial Y_s}{\partial \alpha} + \frac{\lambda+2\mu}{\mu} \frac{\partial}{\partial \alpha}(z X_s) + \frac{\lambda+2\mu}{\mu} \varpi \frac{\partial X_s}{\partial \beta} \\ \iota s M_s &= \frac{\partial Y_s}{\partial \beta} + \frac{\lambda+2\mu}{\mu} \frac{\partial}{\partial \beta}(z X_s) - \frac{\lambda+2\mu}{\mu} \varpi \frac{\partial X_s}{\partial \alpha} \end{aligned}\right\} \ldots\ldots\ldots (9),$$

where Y_s is a function of the same form as X_s, *i.e.* a solution containing $e^{\iota s\phi}$ of equation (3).

To determine the displacements we have to introduce three functions P, Q, R of α and β, defined as follows:

$$P = \tfrac{1}{2}\left(\frac{Q}{\varpi}\frac{\partial \varpi}{\partial \alpha} + \frac{R}{\varpi}\frac{\partial \varpi}{\partial \beta}\right),$$

$Q + \iota R$ is a function of the complex variable $\alpha + \iota\beta$,

Q and R satisfy the equation $\dfrac{\partial}{\partial \alpha}\left(\dfrac{QJ^2}{\varpi^2}\right) + \dfrac{\partial}{\partial \beta}\left(\dfrac{RJ^2}{\varpi^2}\right) = J^2$.

There is no difficulty in determining particular values of Q and R which satisfy the conditions just given, and any values that do so are sufficient for the purpose.

The displacements u, v, w in the directions ϖ, ϕ, z can be expressed in the forms

$$\left.\begin{aligned} u &= U_0 + \sum_1^\infty U_s e^{\iota s\phi} \\ v &= V_0 + \sum_1^\infty V_s e^{\iota s\phi} \\ w &= W_0 + \sum_1^\infty W_s e^{\iota s\phi} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Then we have

$$W_0 = -\int \varpi \frac{\partial Y_0}{\partial \beta} d\alpha - \varpi \frac{\partial Y_0}{\partial \alpha} d\beta \ldots\ldots\ldots\ldots (11),$$

which is the integral of a complete differential in virtue of the differential equation for Y_0.

Also it can be shewn that

$$\left.\begin{aligned} U_0 &= \frac{\partial Z_0}{\partial \alpha} - 4\frac{\partial}{\partial \alpha}\left(PX_0 + Q\frac{\partial X_0}{\partial \alpha} + R\frac{\partial X_0}{\partial \beta}\right) \\ &\quad - \frac{4}{3\varpi}\frac{\partial}{\partial \beta}\left(-PN_0 + Q\frac{\partial N_0}{\partial \alpha} + R\frac{\partial N_0}{\partial \beta}\right) \\ V_0 &= \frac{\partial Z_0}{\partial \beta} - 4\frac{\partial}{\partial \beta}\left(PX_0 + Q\frac{\partial X_0}{\partial \alpha} + R\frac{\partial X_0}{\partial \beta}\right) \\ &\quad + \frac{4}{3\varpi}\frac{\partial}{\partial \alpha}\left(-PN_0 + Q\frac{\partial N_0}{\partial \alpha} + R\frac{\partial N_0}{\partial \beta}\right) \end{aligned}\right\} \quad \ldots\ldots(12),$$

where N_0 is the function defined by (7) and Z_0 is a function of the same form as X_0. Further it can be shewn that

$$\left.\begin{aligned} \iota s U_s &= \frac{\partial W_s}{\partial \alpha} + \varpi M_s \\ \iota s V_s &= \frac{\partial W_s}{\partial \beta} - \varpi L_s \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

and

$$\iota s W_s = Z_s + zY_s + \tfrac{1}{2}\frac{\lambda + 2\mu}{\mu}(\varpi^2 + z^2)X_s + \frac{(\lambda+\mu)s^2 - (\lambda+2\mu)}{\mu(s^2 - \frac{1}{4})}\left(PX_s + Q\frac{\partial X_s}{\partial \alpha} + R\frac{\partial X_s}{\partial \beta}\right) \quad \ldots\ldots(14),$$

where L_s and M_s are defined by (9), and Z_s is a function of the same form as X_s.

The solution is thus expressed in terms of three sets of unknown potential functions X, Y, Z and these can be adapted to satisfy the boundary-conditions. The forms of these functions are known for a few surfaces of revolution such as quadrics, cones, and tores.

206. Plane Strain.

As a further example of the use of curvilinear coordinates we may consider the problem of strain in two dimensions, the position of a point being determined by means of conjugate functions α, β such that

$$\alpha + \iota\beta = f(x + \iota y) \quad \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

Let Δ be the cubical dilatation, and ϖ the elementary rotation

of the medium at any point (α, β); then the equations of equilibrium under no bodily forces are

$$\left.\begin{aligned}(\lambda+2\mu)\frac{\partial\Delta}{\partial\alpha}-2\mu\frac{\partial\varpi}{\partial\beta}&=0\\(\lambda+2\mu)\frac{\partial\Delta}{\partial\beta}+2\mu\frac{\partial\varpi}{\partial\alpha}&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

These are found from (37) of ch. VII. by taking $h_3=1$, $h_1=h_2$, and remembering that $\varpi\,(=\varpi_3)$ is the only one of the components of rotation that occurs.

It is clear from the above equations that $(\lambda+2\mu)\Delta$ and $2\mu\varpi$ are conjugate functions of α and β and therefore also of x and y.

We have next to find the displacements u and v from the equations

$$\left.\begin{aligned}\Delta&=h^2\left\{\frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right)+\frac{\partial}{\partial\beta}\left(\frac{v}{h}\right)\right\}\\2\varpi&=h^2\left\{\frac{\partial}{\partial\alpha}\left(\frac{v}{h}\right)-\frac{\partial}{\partial\beta}\left(\frac{u}{h}\right)\right\}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

in which Δ and 2ϖ are to be regarded as known functions, and h is written for h_1 or h_2.

If we can find any particular solutions of these, then the general solution may be obtained by adding to the particular values of u/h and v/h any others which make Δ and $2\varpi=0$, *i.e.* by taking for the complementary solutions v/h and u/h conjugate functions of α and β, such that $(v+\iota u)/h$ is a function of $\alpha+\iota\beta$.

To obtain the particular solutions we may put

$$\left.\begin{aligned}\frac{u}{h}&=\frac{\partial\phi}{\partial\alpha}-\frac{\partial\psi}{\partial\beta}\\\frac{v}{h}&=\frac{\partial\phi}{\partial\beta}+\frac{\partial\psi}{\partial\alpha}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

and then

$$\left.\begin{aligned}h^2\left(\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2}\right)&=\Delta\\h^2\left(\frac{\partial^2\psi}{\partial\alpha^2}+\frac{\partial^2\psi}{\partial\beta^2}\right)&=2\varpi\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

so that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\Delta,$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=2\varpi,$$

and a particular value of ϕ is the potential of a plane distribution of density $-\Delta/2\pi$, and likewise of ψ for a distribution $-\varpi/\pi$.

This completes the solution in the general case, it will be seen to be arbitrary in two ways viz. $(\lambda+2\mu)\Delta+2\mu\varpi\iota$, and the complementary $(v+\iota u)/h$ are any functions of $\alpha+\iota\beta$.

The above includes as a particular case the theory of solutions in rectangular coordinates x and y. In particular problems it is generally better to use conjugate functions α and β, if it can be arranged that curves $\alpha=$ const. and $\beta=$ const. shall represent the whole of the boundary.

207. Polar Coordinates.

Consider first the case of polar coordinates given by

$$e^{\alpha+\iota\beta}=x+\iota y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

and suppose the bounding surfaces are cylinders of the family α. The forms of Δ and 2ϖ are given by the equations

$$\left.\begin{aligned}\Delta&=\frac{1}{\lambda+2\mu}\Sigma[e^{n\alpha}(A_n\cos n\beta+B_n\sin n\beta)+e^{-n\alpha}(A_n'\cos n\beta+B_n'\sin n\beta)]\\ 2\varpi&=\frac{1}{\mu}\Sigma[e^{n\alpha}(-B_n\cos n\beta+A_n\sin n\beta)+e^{-n\alpha}(B_n'\cos n\beta-A_n'\sin n\beta)]\end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots(21).$$

The value of h is $e^{-\alpha}$; and thus ϕ and ψ have to satisfy the equations

$$\begin{aligned}\frac{\partial^2\phi}{\partial\alpha^2}+\frac{\partial^2\phi}{\partial\beta^2}&=\frac{1}{\lambda+2\mu}\Sigma[e^{(n+2)\alpha}(A_n\cos n\beta+B_n\sin n\beta)\\ &\qquad+e^{-(n-2)\alpha}(A_n'\cos n\beta+B_n'\sin n\beta)],\\ \frac{\partial^2\psi}{\partial\alpha^2}+\frac{\partial^2\psi}{\partial\beta^2}&=\frac{1}{\mu}\Sigma[e^{(n+2)\alpha}(-B_n\cos n\beta+A_n\sin n\beta)\\ &\qquad+e^{-(n-2)\alpha}(B_n'\cos n\beta-A_n'\sin n\beta)].\end{aligned}$$

Particular integrals of these equations are

$$\begin{aligned}\phi&=\frac{1}{\lambda+2\mu}\Sigma\left[\frac{1}{4(n+1)}e^{(n+2)\alpha}(A_n\cos n\beta+B_n\sin n\beta)\right.\\ &\qquad\left.-\frac{1}{4(n-1)}e^{-(n-2)\alpha}(A_n'\cos n\beta+B_n'\sin n\beta)\right],\\ \psi&=\frac{1}{\mu}\Sigma\left[\frac{1}{4(n+1)}e^{(n+2)\alpha}(-B_n\cos n\beta+A_n\sin n\beta)\right.\\ &\qquad\left.-\frac{1}{4(n-1)}e^{-(n-2)\alpha}(B_n'\cos n\beta-A_n'\sin n\beta)\right];\end{aligned}$$

and particular integrals of equations (17) are therefore

$$ue^{\alpha} = \Sigma \left[\frac{1}{4(n+1)} \left(\frac{n+2}{\lambda+2\mu} - \frac{n}{\mu} \right) e^{(n+2)\alpha} (A_n \cos n\beta + B_n \sin n\beta) \right.$$
$$\left. + \frac{1}{4(n-1)} \left(\frac{n-2}{\lambda+2\mu} - \frac{n}{\mu} \right) e^{-(n-2)\alpha} (A_n' \cos n\beta + B_n' \sin n\beta) \right] \ldots (22),$$

and

$$ve^{\alpha} = \Sigma \left[\frac{1}{4(n+1)} \left(\frac{n+2}{\mu} - \frac{n}{\lambda+2\mu} \right) e^{(n+2)\alpha} (-B_n \cos n\beta + A_n \sin n\beta) \right.$$
$$\left. + \frac{1}{4(n-1)} \left(\frac{n-2}{\mu} - \frac{n}{\lambda+2\mu} \right) e^{-(n-2)\alpha} (B_n' \cos n\beta - A_n' \sin n\beta) \right]$$
$$\ldots\ldots\ldots\ldots (23).$$

We have to add to these complementary solutions of the forms

$$\left. \begin{aligned} ue^{\alpha} &= \Sigma \left[e^{n\alpha} (-D_n \cos n\beta + C_n \sin n\beta) + e^{-n\alpha} (D_n' \cos n\beta - C_n' \sin n\beta) \right] \\ ve^{\alpha} &= \Sigma \left[e^{n\alpha} (\ C_n \cos n\beta + D_n \sin n\beta) + e^{-n\alpha} (C_n' \cos n\beta + D_n' \sin n\beta) \right] \end{aligned} \right\}$$
$$\ldots\ldots\ldots\ldots (24).$$

The tangential and normal tractions F and G across any cylindrical surface of the family α are, by (38) of ch. VII,

$$F = \lambda\Delta + 2\mu e^{-\alpha} \frac{\partial u}{\partial \alpha},$$

$$G = -2\mu\varpi + 2\mu e^{-\alpha} \frac{\partial v}{\partial \alpha};$$

and the values of these are easily written down.

In general there is also a traction $\lambda\Delta$ perpendicular to the plane of (α, β).

208. Failing Cases.

The general formulæ fail when $n = 0$ or 1. In the first of these cases we may consider separately the solutions in which Δ is constant and those in which ϖ is constant.

When ϖ is constant and Δ is zero we have $\phi = 0$ and $\psi = \frac{1}{2}\varpi e^{2\alpha}$, also $u = 0$ and $v = \varpi e^{\alpha} + Ae^{-\alpha}$. The tractions F and G are $F = 0$, and $G = -2\mu A e^{-2\alpha}$. If any cylinder be free we must have $A = 0$. This corresponds to the torsion of a cylindrical shaft, and the strain at any point is the same whether there be a coaxal cylindrical cavity or the shaft be complete up to the axis.

When Δ is constant and ϖ is zero we have $\phi = \frac{1}{4}\Delta e^{2\alpha}$ and $\psi = 0$, also $u = \frac{1}{2}\Delta e^{\alpha} + Ae^{-\alpha}$ and $v = 0$. The tractions F and G are

$$F = (\lambda + \mu)\Delta - 2\mu A e^{-2\alpha}, \quad G = 0.$$

If there be a cylindrical cavity in an infinite solid and the displacement at infinity be $-ce^{\alpha}$ towards the axis of the cavity, we have $v=0$ and

$$u=-c\left(r+\frac{\lambda+\mu}{\mu}\frac{a^2}{r}\right),$$

where a is the radius of the inner boundary, and r is the distance e^{α} of any point from the axis. This corresponds to the case of a bar under extension when there is a cylindrical cavity parallel to the axis of the bar, the distance of the cavity from the axis of the bar being large compared with the diameter of the cavity and small compared with the diameter of the bar. The radial strain in the neighbourhood of the cavity becomes an extension equal to λ/μ times the radial contraction that would have place if there were no cavity.

The failure in the case of $n=1$ is caused by the occurrence of $(n-1)$ in the denominator of (22) and (23). In order to find the forms applicable to this case we may start by supposing

$$\Delta=\frac{1}{\lambda+2\mu}e^{-\alpha}(A\cos\beta+B\sin\beta),$$

$$2\varpi=\frac{1}{\mu}e^{-\alpha}(B\cos\beta-A\sin\beta).$$

Then we have

$$\frac{\Delta}{h^2}=\frac{1}{\lambda+2\mu}e^{\alpha}(A\cos\beta+B\sin\beta),$$

$$\frac{2\varpi}{h^2}=\frac{1}{\mu}e^{\alpha}(B\cos\beta-A\sin\beta);$$

and the functions ϕ and ψ are given by

$$\phi=\tfrac{1}{2}\frac{1}{\lambda+2\mu}\alpha e^{\alpha}(A\cos\beta+B\sin\beta),$$

$$\psi=\frac{1}{2\mu}\alpha e^{\alpha}(B\cos\beta-A\sin\beta).$$

Thus the particular integrals for u and v, so far as they depend upon these terms, are

$$\left.\begin{aligned}u&=\left(\tfrac{1}{2}\frac{1+\alpha}{\lambda+2\mu}+\frac{\alpha}{2\mu}\right)(A\cos\beta+B\sin\beta)\\ v&=\left(\tfrac{1}{2}\frac{\alpha}{\lambda+2\mu}+\frac{1+\alpha}{2\mu}\right)(B\cos\beta-A\sin\beta)\end{aligned}\right\}\quad\ldots\ldots(25).$$

209. General Formulæ.

Now taking ordinary polar coordinates r and θ so that $e^{\alpha} = r$, $\beta = \theta$, we have for the general forms of u and v

$$u = \left(\tfrac{1}{2}\frac{1+\log r}{\lambda+2\mu} + \frac{\log r}{2\mu}\right)(A_1' \cos\theta + B_1' \sin\theta)$$
$$+ \sum_1^\infty \left(\frac{n+2}{\lambda+2\mu} - \frac{n}{\mu}\right)\frac{r^{n+1}}{4(n+1)}(A_n \cos n\theta + B_n \sin n\theta)$$
$$+ \sum_2^\infty \left(\frac{n-2}{\lambda+2\mu} - \frac{n}{\mu}\right)\frac{r^{-n+1}}{4(n-1)}(A_n' \cos n\theta + B_n' \sin n\theta)$$
$$+ \sum_1^\infty [r^{n-1}(-D_n \cos n\theta + C_n \sin n\theta) + r^{-(n+1)}(D_n' \cos n\theta - C_n' \sin n\theta)]$$
$$\ldots\ldots\ldots\ldots(26),$$

and

$$v = \left(\tfrac{1}{2}\frac{\log r}{\lambda+2\mu} + \frac{1+\log r}{2\mu}\right)(B_1' \cos\theta - A_1' \sin\theta)$$
$$+ \sum_1^\infty \left(\frac{n+2}{\mu} - \frac{n}{\lambda+2\mu}\right)\frac{r^{n+1}}{4(n+1)}(-B_n \cos n\theta + A_n \sin n\theta)$$
$$+ \sum_2^\infty \left(\frac{n-2}{\mu} - \frac{n}{\lambda+2\mu}\right)\frac{r^{-n+1}}{4(n-1)}(B_n' \cos n\theta - A_n' \sin n\theta)$$
$$+ \sum_1^\infty [r^{n-1}(C_n \cos n\theta + D_n \sin n\theta) + r^{-(n+1)}(C_n' \cos n\theta + D_n' \sin n\theta)]$$
$$\ldots\ldots\ldots\ldots(27).$$

In the same notation we have

$$\left.\begin{aligned}\Delta &= \frac{1}{\lambda+2\mu}\sum_1^\infty [r^n(A_n \cos n\theta + B_n \sin n\theta) + r^{-n}(A_n' \cos n\theta + B_n' \sin n\theta)] \\ 2\varpi &= \frac{1}{\mu}\sum_1^\infty [r^n(-B_n \cos n\theta + A_n \sin n\theta) + r^{-n}(B_n' \cos n\theta - A_n' \sin n\theta)]\end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots(28);$$

and the tractions at a cylindrical surface $r = \text{const.}$ are given by

$$\left.\begin{aligned}F &= \lambda\Delta + 2\mu\frac{\partial u}{\partial r} \\ G &= -2\mu\varpi + 2\mu\frac{\partial v}{\partial r}\end{aligned}\right\} \ldots\ldots\ldots\ldots(29).$$

This gives means for the complete analytical solution of any problem of plane strain in a solid bounded by coaxal circular cylinders.

210. Particular Example.

As an example we may consider the case where there is a cylindrical cavity of radius a in an infinite solid, and at an infinite

distance there is a distribution of shear. To represent this we may take the displacements referred to the system of fixed axes of x and y to be U and V, and suppose that at an infinite distance

$$U = sy, \quad V = 0.$$

In the notation of the last article the conditions at infinity become

$$u\cos\theta - v\sin\theta = sr\sin\theta,$$
$$u\sin\theta + v\cos\theta = 0;$$

or

$$u = \tfrac{1}{2}sr\sin 2\theta,$$
$$v = \tfrac{1}{2}sr\cos 2\theta - \tfrac{1}{2}sr.$$

We have already seen that $u = 0$, $v = -\frac{1}{2}sr$ satisfy the equations, and make the tractions F and G vanish at every cylindrical surface, so that we shall have to add this solution to the solution for

$$u = \tfrac{1}{2}sr\sin 2\theta, \quad v = \tfrac{1}{2}sr\cos 2\theta,$$

when r is very great.

From the general solutions (26) and (27) we have to keep the terms in B_2', C_2, C_2'. To satisfy the conditions when r is very great we have to take $C_2 = \frac{1}{2}s$. The condition that there is no traction across the surface $r = a$ gives two relations among the three constants by which B_2' and C_2' are determined. The work may be left to the reader, and the result is that

$$\left.\begin{aligned} u &= \left(\frac{\lambda+2\mu}{\lambda+\mu}\frac{a^2}{r} + \tfrac{1}{2}r - \tfrac{1}{2}\frac{a^4}{r^3}\right)s\sin 2\theta, \\ v &= \left(\frac{\mu}{\lambda+\mu}\frac{a^2}{r} + \tfrac{1}{2}r + \tfrac{1}{2}\frac{a^4}{r^3}\right)s\cos 2\theta - \tfrac{1}{2}sr \end{aligned}\right\} \ldots\ldots (30).$$

It may be as well perhaps to remark that this problem does not, like the corresponding one in art. 169, yield a result in connexion with the theory of torsion. In the case of torsion a very important part of the shear consists as we know of a shifting of the fibres of the twisted prism parallel to the axis of the prism, and our work above, being confined to displacements in one plane, does not take this into account.

211. Elliptic Coordinates.

We shall next consider the case of elliptic coordinates given by

$$x + \iota y = c\cosh(\alpha + \iota\beta) \ldots\ldots\ldots\ldots\ldots (31),$$

and suppose in the first place that the elastic medium extends to infinity, and is bounded internally by an elliptic cylindrical surface

of the family α, say $\alpha = \alpha_0$, which is deformed in a given manner. Then, according to art. 206, we have to take for Δ and 2ϖ series of the form

$$\Delta = \Sigma A_n e^{-n\alpha} e^{\iota n\beta},$$

$$2\varpi\iota = \frac{\lambda + 2\mu}{\mu} \Sigma A_n e^{-n\alpha} e^{\iota n\beta},$$

in which A_n is a complex constant; and we may at the end keep only the real part of the solution.

Now the displacements have to be found from the equations

$$\frac{\Delta}{h^2} = \frac{\partial}{\partial\alpha}\left(\frac{u}{h}\right) + \frac{\partial}{\partial\beta}\left(\frac{v}{h}\right),$$

$$\frac{2\varpi}{h^2} = \frac{\partial}{\partial\alpha}\left(\frac{v}{h}\right) - \frac{\partial}{\partial\beta}\left(\frac{u}{h}\right),$$

where $$h^{-2} = \tfrac{1}{2}c^2(\cosh 2\alpha - \cos 2\beta).$$

The functions Δh^{-2}, $2\varpi h^{-2}$ can each be expressed as sums of terms of the forms

$$\left\{e^{-(n+2)\alpha}\right\} \begin{matrix}\cos n\beta \\ \sin n\beta\end{matrix}, \text{ and } \left\{e^{-n\alpha}\right\} \begin{matrix}\cos(n+2)\beta \\ \sin(n+2)\beta\end{matrix},$$

and the equations for u and v can clearly be satisfied by assuming for u and v sums of terms of these forms with suitable coefficients These are the particular integrals of the equations for the displacements, and the complementary functions will be found by taking $(v + \iota u)/h$ any function of $(\alpha + \iota\beta)$ and therefore by taking for v/h and u/h functions of the same forms as Δ and $2\mu\varpi/(\lambda + 2\mu)$.

Now suppose definitely that

$$\left.\begin{aligned} \Delta &= \frac{4}{c^2(\lambda + 2\mu)} \Sigma e^{-n\alpha}(A_n \cos n\beta + B_n \sin n\beta) \\ 2\varpi &= \frac{4}{c^2\mu} \Sigma e^{-n\alpha}(B_n \cos n\beta - A_n \sin n\beta) \end{aligned}\right\} \quad \ldots\ldots(32);$$

then we can easily verify that

$$\left.\begin{aligned} (\lambda + 2\mu)\frac{\Delta}{h^2} &= \Sigma\left[e^{-(n+2)\alpha}\{(A_n - A_{n+2})\cos n\beta + (B_n - B_{n+2})\sin n\beta\}\right. \\ &\quad \left. + e^{-(n-2)\alpha}\{(A_n - A_{n-2})\cos n\beta + (B_n - B_{n-2})\sin n\beta\}\right] \\ 2\mu\frac{\varpi}{h^2} &= \Sigma\left[e^{-(n+2)\alpha}\{(B_n - B_{n+2})\cos n\beta - (A_n - A_{n+2})\sin n\beta\}\right. \\ &\quad \left. + e^{-(n-2)\alpha}\{(B_n - B_{n-2})\cos n\beta - (A_n - A_{n-2})\sin n\beta\}\right] \end{aligned}\right\}$$

$$\ldots\ldots(33);$$

and again we can easily verify that u/h and v/h are given by

$$\frac{u}{h} = \Sigma \frac{e^{-(n+2)\alpha}}{4(n+1)}\left(\frac{n}{\mu} - \frac{n+2}{\lambda+2\mu}\right)\left\{(A_n - A_{n+2})\cos n\beta + (B_n - B_{n+2})\sin n\beta\right\}$$
$$- \Sigma \frac{e^{-(n-2)\alpha}}{4(n-1)}\left(\frac{n}{\mu} - \frac{n-2}{\lambda+2\mu}\right)\left\{(A_n - A_{n-2})\cos n\beta + (B_n - B_{n-2})\sin n\beta\right\}$$
$$+ \Sigma e^{-n\alpha}(D_n \cos n\beta - C_n \sin n\beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34),$$

$$\frac{v}{h} = -\Sigma \frac{e^{-(n+2)\alpha}}{4(n+1)}\left(\frac{n+2}{\mu} - \frac{n}{\lambda+2\mu}\right)\left\{(B_n - B_{n+2})\cos n\beta - (A_n - A_{n+2})\sin n\beta\right\}$$
$$+ \Sigma \frac{e^{-(n-2)\alpha}}{4(n-1)}\left(\frac{n-2}{\mu} - \frac{n}{\lambda+2\mu}\right)\left\{(B_n - B_{n-2})\cos n\beta - (A_n - A_{n-2})\sin n\beta\right\}$$
$$+ \Sigma e^{-n\alpha}(C_n \cos n\beta + D_n \sin n\beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35).$$

Suppose the boundary-conditions given in the form

$$\left.\begin{aligned} \frac{u}{h} &= \Sigma e^{-n\alpha}(L_n \cos n\beta + M_n \sin n\beta) \\ \frac{v}{h} &= \Sigma e^{-n\alpha}(M_n' \cos n\beta + L_n' \sin n\beta) \end{aligned}\right\} \quad \ldots\ldots\ldots (36)$$

when $\alpha = \alpha_0$. By equating the coefficients of $\cos n\beta$ and $\sin n\beta$ we get four equations to determine the four sets of constants A_n, B_n, C_n, D_n.

These equations are

$$\left.\begin{aligned} &\frac{e^{-2\alpha_0}}{4(n+1)}\left(\frac{n}{\mu} - \frac{n+2}{\lambda+2\mu}\right)(A_n - A_{n+2}) \\ &\qquad - \frac{e^{2\alpha_0}}{4(n-1)}\left(\frac{n}{\mu} - \frac{n-2}{\lambda+2\mu}\right)(A_n - A_{n-2}) + D_n = L_n \\ &\frac{e^{-2\alpha_0}}{4(n+1)}\left(\frac{n+2}{\mu} - \frac{n}{\lambda+2\mu}\right)(A_n - A_{n+2}) \\ &\qquad - \frac{e^{2\alpha_0}}{4(n-1)}\left(\frac{n-2}{\mu} - \frac{n}{\lambda+2\mu}\right)(A_n - A_{n-2}) + D_n = L_n' \end{aligned}\right\} \quad \ldots\ldots\ldots (37)$$

and

$$\left.\begin{aligned} &\frac{e^{-2\alpha_0}}{4(n+1)}\left(\frac{n}{\mu} - \frac{n+2}{\lambda+2\mu}\right)(B_n - B_{n+2}) \\ &\qquad - \frac{e^{2\alpha_0}}{4(n-1)}\left(\frac{n}{\mu} - \frac{n-2}{\lambda+2\mu}\right)(B_n - B_{n-2}) - C_n = M_n \\ &\frac{e^{-2\alpha_0}}{4(n+1)}\left(\frac{n+2}{\mu} - \frac{n}{\lambda+2\mu}\right)(B_n - B_{n+2}) \\ &\qquad - \frac{e^{2\alpha_0}}{4(n-1)}\left(\frac{n-2}{\mu} - \frac{n}{\lambda+2\mu}\right)(B_n - B_{n-2}) - C_n = -M_n' \end{aligned}\right\} \quad \ldots\ldots\ldots (38).$$

From the first two we get a difference-equation for the A's, and from the second two we get a difference-equation for the B's. When these are solved D_n and C_n are given by one of (37) and one of (38).

As an example[1] suppose the cylinder α_0, whose principal semi-axes are a and b, turned through a small angle ϕ. The boundary-conditions are

$$\frac{u}{h} = e^{-2\alpha_0} \frac{(a+b)^2}{2} \phi \sin 2\beta, \quad \frac{v}{h} = ab\phi, \quad \text{when } \alpha = \alpha_0.$$

All the A's vanish, B_0 and all the odd B's vanish, and $B_2 = B_4 = \ldots$, all the D's vanish and all the C's except C_0 and C_2, and we find

$$C_2 = -\tfrac{1}{2}(a+b)^2 \frac{\mu}{\lambda + 3\mu} \phi,$$

$$B_2 = 2C_2 e^{-2\alpha_0} (\lambda + 2\mu),$$

$$C_0 = ab\phi + \tfrac{1}{2}(a+b)^2 \phi e^{-4\alpha_0} \frac{\lambda + 2\mu}{\lambda + 3\mu};$$

and $$\frac{u}{h} = \tfrac{1}{2}(a+b)^2 \frac{\mu}{\lambda + 3\mu} \phi \left(e^{-2\alpha} + \frac{\lambda + 2\mu}{\mu} e^{-2\alpha_0} \right) \sin 2\beta,$$

$$\frac{v}{h} = ab\phi + \tfrac{1}{2}(a+b)^2 \phi (e^{-2\alpha_0} - e^{-2\alpha}) \frac{(\lambda + 2\mu) e^{-2\alpha_0} + \mu \cos 2\beta}{\lambda + 3\mu}.$$

It appears that at a very great distance the displacements of points on a confocal cylinder vanish, since h vanishes. The cubical dilatation of the medium is

$$\frac{4B_2}{c^2(\lambda + 2\mu)} \sum_{m=1}^{m=\infty} e^{-2m\alpha} \sin 2m\beta,$$

and the rotation of the medium is

$$\frac{4B_2}{c^2\mu} \sum_{m=1}^{m=\infty} e^{-2m\alpha} \cos 2m\beta,$$

which vanish at an infinite distance.

[1] This example was suggested by Mr Webb.

The corresponding problem of displacement within a cylinder due to a rotation of its boundary is much simpler.

Consider a solution in which $\Delta = 0$, and ϖ is constant and equal to ϕ, we have

$$\frac{2\varpi}{h^2} = c^2\phi\,(\cosh 2\alpha - \cos 2\beta)$$

$$= \tfrac{1}{2}c^2\phi\left[\frac{\partial}{\partial\alpha}(\sinh 2\alpha) - \frac{\partial}{\partial\beta}(\sin 2\beta)\right],$$

so that $v/h = \tfrac{1}{2}c^2\phi \sinh 2\alpha,\ u/h = \tfrac{1}{2}c^2\phi \sin 2\beta.$

When $\alpha = \alpha_0$ we find

$$v/h = ab\phi,\ u/h = \tfrac{1}{2}e^{-2\alpha_0}(a+b)^2\,\phi \sin 2\beta.$$

Thus the above solution satisfies all the conditions.

NOTES.

Note A. On Shear and Shearing Stress.

The term "shear" was first used by engineers to denote tangential stress, and is so used in Rankine's *Applied Mechanics*. The usage of it for sliding strain in this work might be justified by reference to Sir W. Thomson, now Lord Kelvin, and many other eminent authorities, theoretical and practical. The kind of strain called shear has been considered in ch. I, and the kind of stress called shearing stress has been considered in ch. II. The object of this note is to insist more fully than is done in those chapters on the twofold character of both shear and shearing stress as they occur in the mathematical expressions. For simplicity we shall limit our consideration to the case of infinitesimal displacements.

The shears are represented by such expressions as $\partial w/\partial y + \partial v/\partial z$. Now this expression is the sum of two simple shears, viz.: a simple shear $\partial w/\partial y$ of the planes $y = \text{const.}$ parallel to the axis z, and a simple shear $\partial v/\partial z$ of the planes $z = \text{const.}$ parallel to the axis y. In like manner if we define the (infinitesimal) shear of two initially rectangular lines (1) and (2) to be the cosine of the angle between them after strain—a definition which has been shewn to coincide with the definition in terms of sliding motion—then this shear will be made up of a simple shear parallel to (2) of the planes perpendicular to (1), and a simple shear parallel to (1) of the planes perpendicular to (2). The shears that occur in mathematical expressions are in fact generally the sums of two such simple shears which are not at first separated. Thus in the energy-function the terms in a for example are just the same whatever be the proportion in a of the simple shear parallel to y to that parallel to z.

Shearing stress also is of a twofold character, but the like ambiguity does not occur. Shearing stress consists of tangential stresses across two perpendicular planes, but these are always *equal*.

We know that a simple shear c is equivalent to equal extension and contraction each $\frac{1}{2}c$, and conversely that equal extension and contraction each e are equivalent to a simple shear of amount $2e$, and in the same way the extension and contraction might be taken to be equivalent to *two* simple shears each of amount e, which combine in the manner explained above;

or again the same extension and contraction will be the equivalents of two simple shears whose sum is $2e$ and whose ratio is anything whatever.

Equal pressure and tension each P are in like manner equivalent to a shearing stress, but the amount of the shearing stress is P. This shearing stress is really a stress-system consisting of equal tangential stresses P on two perpendicular planes.

The above remarks appear to contain the secret of the "discrepant reckonings of shear and shearing stress" to which Lord Kelvin has frequently called attention. (See *e.g.* Thomson and Tait's *Nat. Phil.* Part II. art. 681, and *Lectures on Molecular Dynamics* p. 176.) The discrepancy appears to arise from the combination in a shear of two simple shears whose ratio it is unnecessary to know, while the tangential stresses combined in a shearing stress are always equal. Writing the discrepant statements in parallel columns we have

Equal extension and contraction each e are equivalent to two simple shears of perpendicular planes; the sum of the shears is $2e$ and their ratio may be anything whatever.	Equal pressure and tension each P are equivalent to tangential stresses on two perpendicular planes; each of these is of amount P.

Finally we may note that the values of the two simple shears will be equal if the strain be pure. It follows that, if we regard any small strain as analysed into a small rotation and a small pure strain, then the extensions and contractions to which the pure shears are equivalent are always obtained from the simple shears by precisely the same rule as that by which the pressures and tensions are obtained from the tangential stresses.

Note B. On Æolotropic Bodies.

Æolotropy has been defined in art. 24 as variability of the physical character of a body depending on directions fixed with reference to the body. Fibrous and luminated bodies as well as crystals exhibit such variability of elastic character, and in regard to other physical properties (optical, magnetic, thermal &c.) such variability is exhibited by many well-known crystalline bodies. The theory of elastic crystals given in the text takes account of elastic properties only. This theory is not proved, and it is not here suggested that, even supposing it proved for elastic properties, it would hold for other physical properties. In other words it is not suggested that the æolotropy of a body for the transmission of light waves (for example) is similar to its æolotropy for elastic reactions.

The theory connects elastic quality with crystallographic form; and it leads, in the case of each crystal form, to a certain number of elastic constants. In the absence of definite experimental evidence the assumption that the maximum number of these constants for a given body, and the way they enter into the stress-strain relations, are correctly given appears to have considerable probability. I think it will be generally admitted that a spherical portion of a cubic crystal, for example, would exhibit identity of physical

properties after rotation through 90^0 about any one of the crystallographic axes. It may however be questioned whether the constants given by the theory are really independent. In other words I think it will be generally admitted that crystalline bodies are at least as nearly isotropic as the theory makes them, but it may be questioned whether they are not more nearly isotropic. Optical experiments appear in some cases to favour an affirmative answer to this question.

Taking again the case of cubic crystals, it is easy to shew that the rigidity (art. 42) for two directions in a principal plane of symmetry, making half right angles with the two principal axes of symmetry that lie in the plane, is $\frac{1}{2}(a_{11}-a_{12})$, while the rigidity for two principal axes of symmetry is a_{44}.

This is the property which Lord Kelvin has noted as characteristic of "cubic asymmetry" or "cyboïd æolotropy", and he has, on optical grounds, questioned the existence of bodies possessing the property. (*Lectures on Molecular Dynamics* p. 158.) The experiments of Prof. Voigt (art. 45) appear to shew that $\frac{1}{2}(a_{11}-a_{12})$ and a_{44} have, for some well-known cubic crystals, widely different values.

With regard to cubic crystals it may be as well to notice further two points:

(*a*) That if the luminiferous ether in any body were similar in elastic quality to the elastic cubic crystals discussed in art. 37 the body would be doubly refracting and would exhibit conical refraction, but the wave-surface would be much more complicated than Fresnel's.

(*b*) That although the three principal Young's moduluses, the three principal rigidities, and the three principal Poisson's ratios are equal, such bodies are not "transversely isotropic".

With regard to "transverse isotropy" it may be noticed that a body cannot be transversely isotropic in the plane (x, y) unless its energy-function reduce to the form for hexagonal crystals, viz:

$$A(e+f)^2+Cg^2+2F(e+f)g+N(c^2-4ef)+L(a^2+b^2).$$

For example a tetragonal crystal is not transversely isotropic although it has two principal Young's moduluses, two principal rigidities, and two principal Poisson's ratios equal.

Note C. On Betti's Method of Integration.

Mr Larmor suggests to me that the analysis in arts. 141, 142 admits of a physical interpretation.

Suppose a small spherical element of a solid whose centre is a given point is uniformly extended. If the solid be unlimited and under no bodily force, the displacements at any point can be shewn to be proportional to $\partial r^{-1}/\partial x$, $\partial r^{-1}/\partial y$, $\partial r^{-1}/\partial z$. If the solid be limited by a *free* surface certain displacements will take place at the surface. If the surface be fixed certain tractions will have to be applied to the surface. The interpretation to be made involves the displacements that exist when the surface is free and the spherical element

about a given point is extended, and the surface-tractions that must be applied to hold the surface fixed when the same state of dilatation is produced in the spherical element.

Equation (40) on p. 244 shews that the dilatation produced at any point by a given system of surface-displacements is proportional to the work done by the tractions that must be applied to hold the surface fixed, when there is dilatation of the spherical element about the point, acting through the given surface-displacements; and equation (41) on the same page shews that the dilatation produced at any point by a given system of surface-tractions is proportional to the work done by these tractions acting through the displacements that take place when the surface is free and there is dilatation of the spherical element about the point.

There is a like interpretation of such equations as (48) and (46) on p. 246 for rotation about any given line in terms of the tractions that must be applied to hold the surface fixed when a spherical element about a given point is made to rotate about the line, and of the displacements that take place when the surface is free and a similar rotation is effected at the point. In fact in the above statements we have merely to read 'rotation about a given line' for 'dilatation'.

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